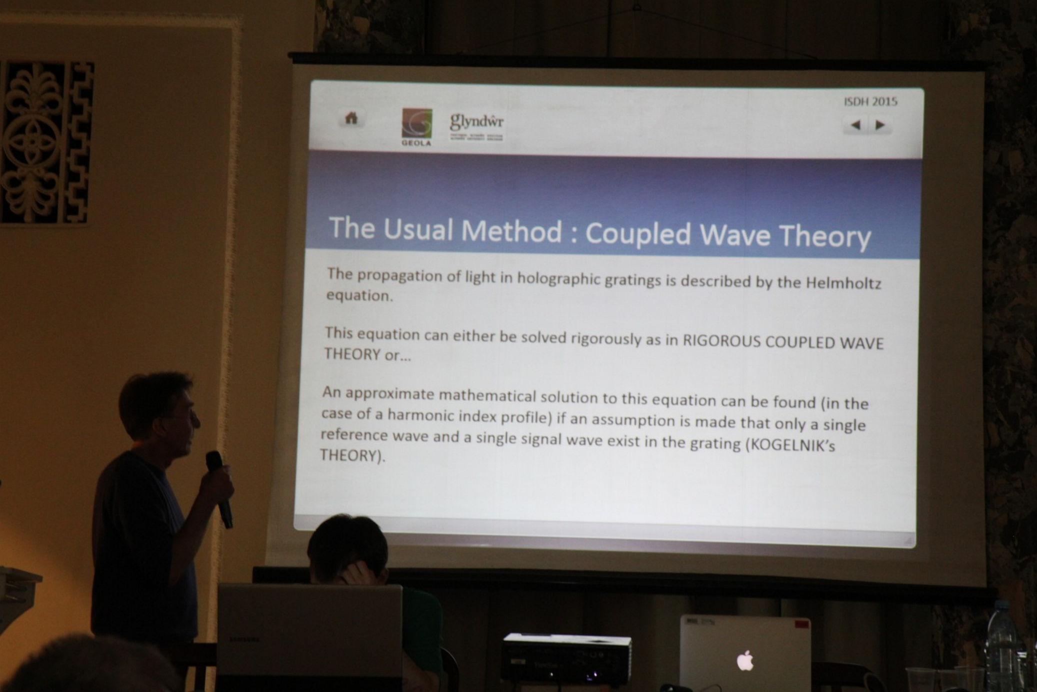
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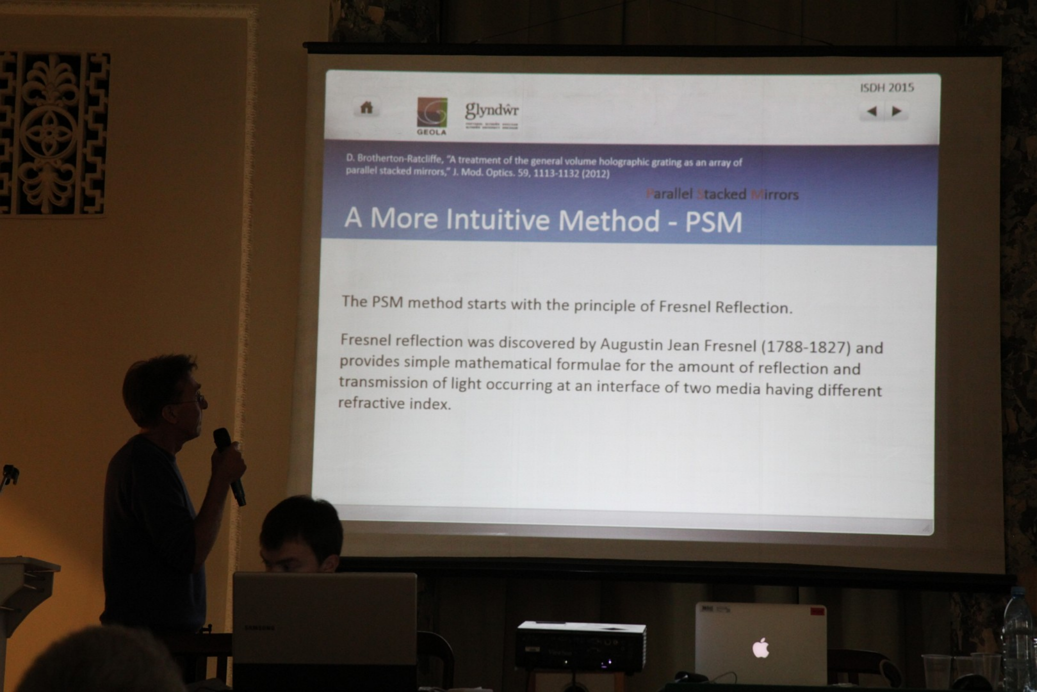
The Usual Method : Coupled Wave Theory
The propagation of light in holographic gratings is described by the Helmholtz equation.
This equation can either be solved rigorously as in RIGOROUS COUPLED WAVE THEORY or...
An approximate mathematical solution to this equation can be found (in the case of a harmonic index profile) if an assumption is made that only a single reference wave and a single signal wave exist in the grating (KOGELNIK's THEORY).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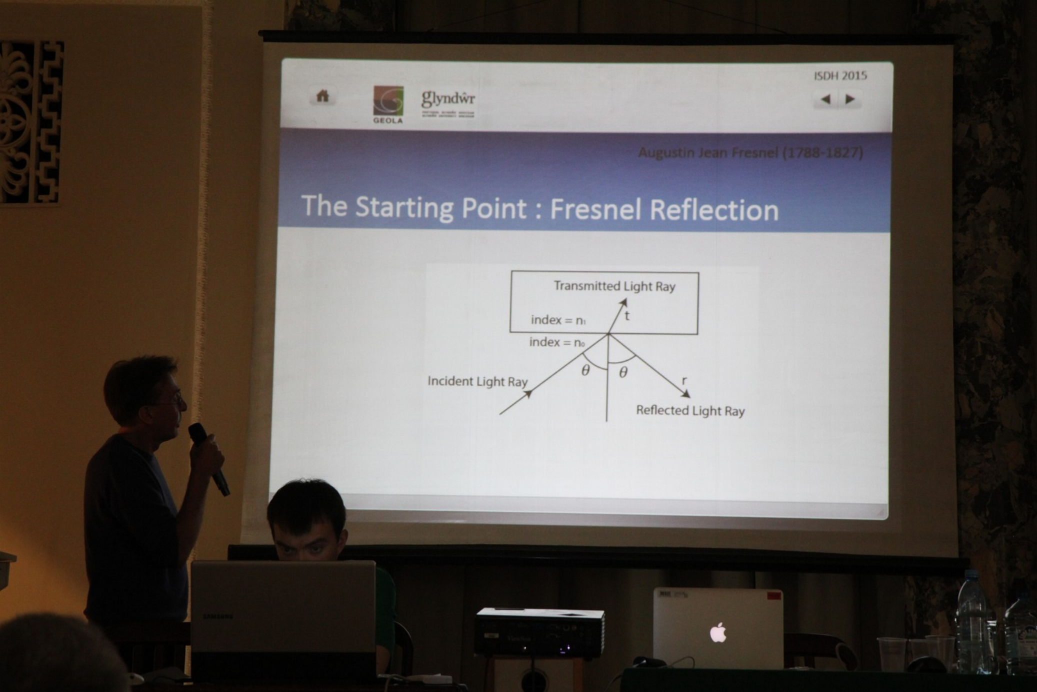
D. Brotherton-Ratcliffe, "A treatment of the general volume holographic grating as an array of parallel stacked mirrors," J. Mod. Optics. 59, 1113-1132 (2012)
Parallel Stacked Mirrors
A More Intuitive Method - PSM
The PSM method starts with the principle of Fresnel Reflection.
Fresnel reflection was discovered by Augustin Jean Fresnel (1788-1827) and provides simple mathematical formulae for the amount of reflection and transmission of light occurring at an interface of two media having different refractive index.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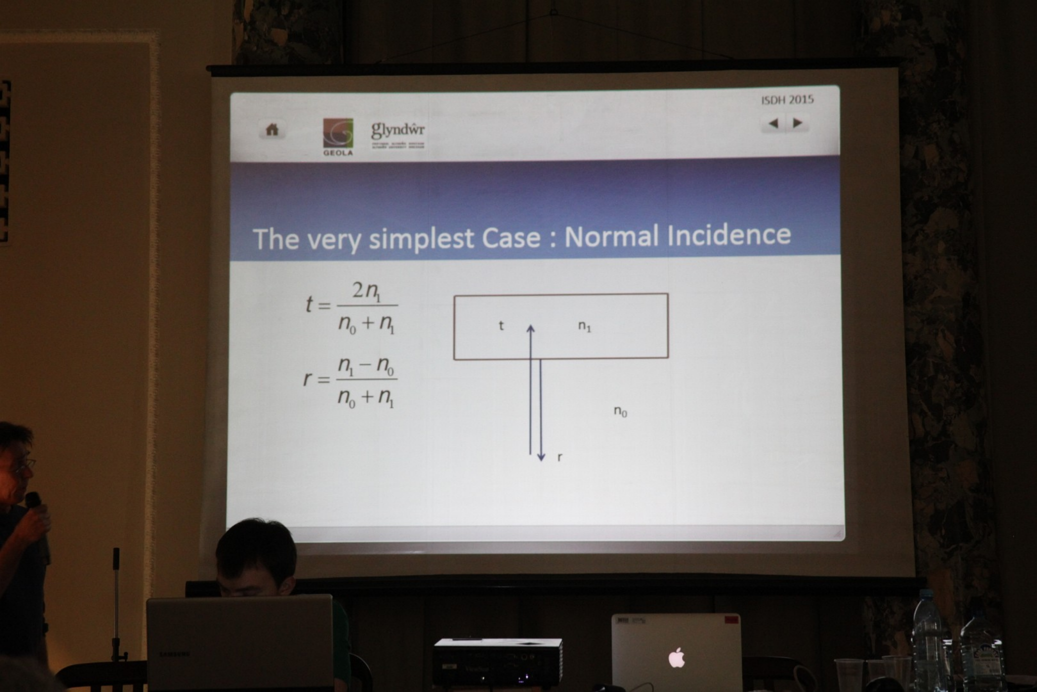
Augustin Jean Fresnel (1788-1827)
The Starting Point : Fresnel Reflection
Transmitted Light Ray
t
index = n1
index = n0
θ
θ
Incident Light Ray
r
Reflected Light Ray
SAMSUNG



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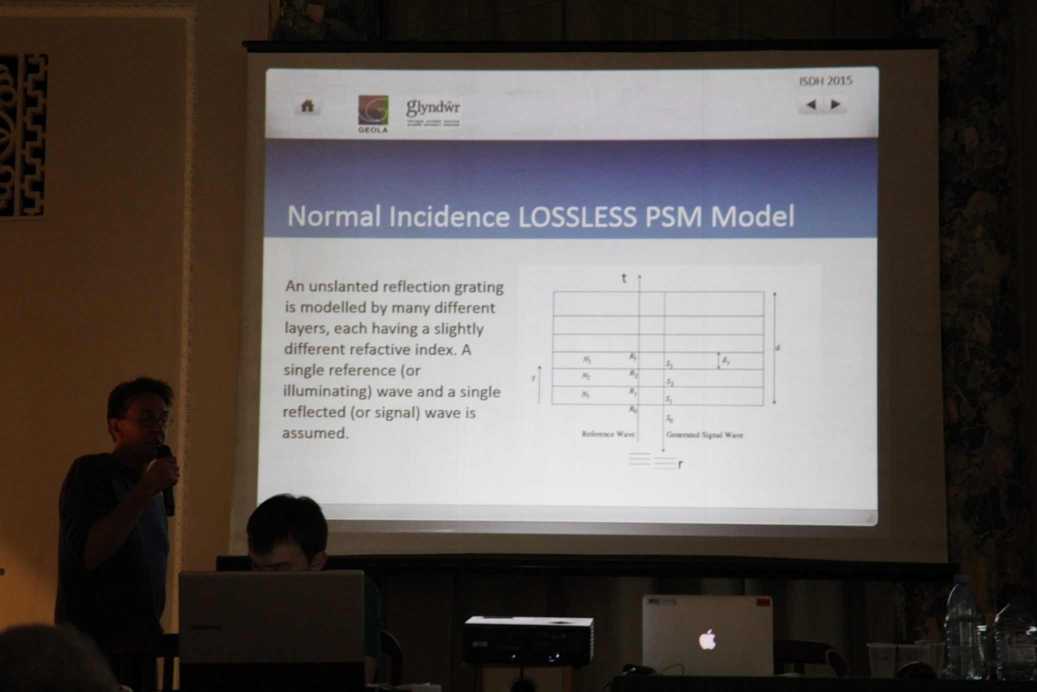
The very simplest Case : Normal Incidence
$$t = \frac{2n_1}{n_0 + n_1}$$
$$r = \frac{n_1 - n_0}{n_0 + n_1}$$
t
n_1
n_0
r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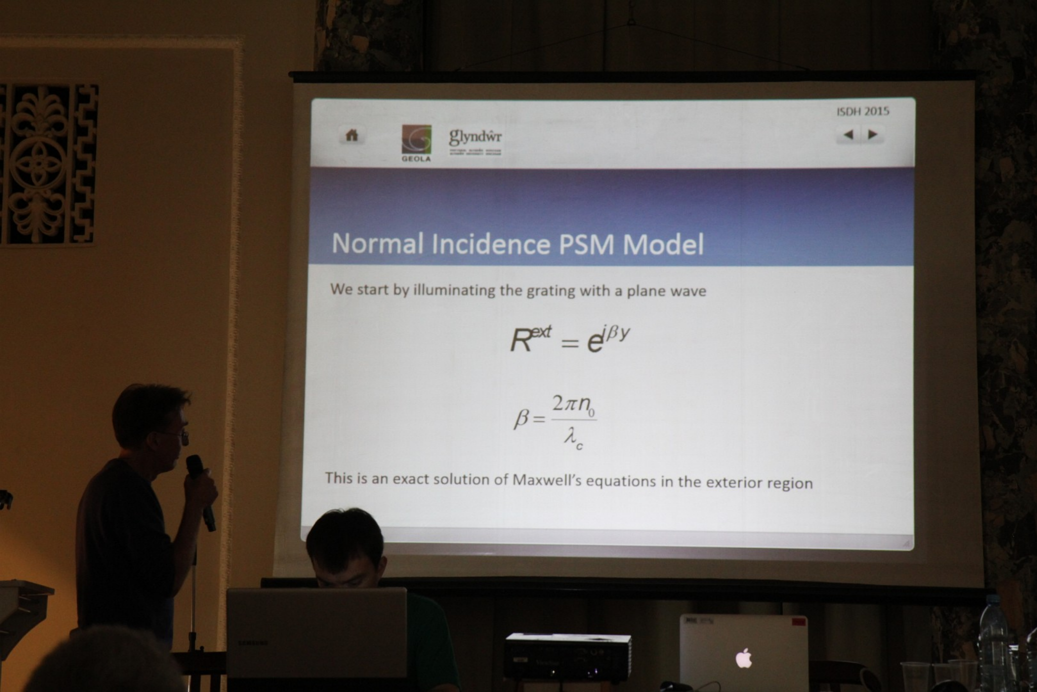
Normal Incidence LOSSLESS PSM Model
An unslanted reflection grating is modelled by many different layers, each having a slightly different refactive index. A single reference (or illuminating) wave and a single reflected (or signal) wave is assumed.
t
d
y
Reference Wave
Generated Signal Wave
r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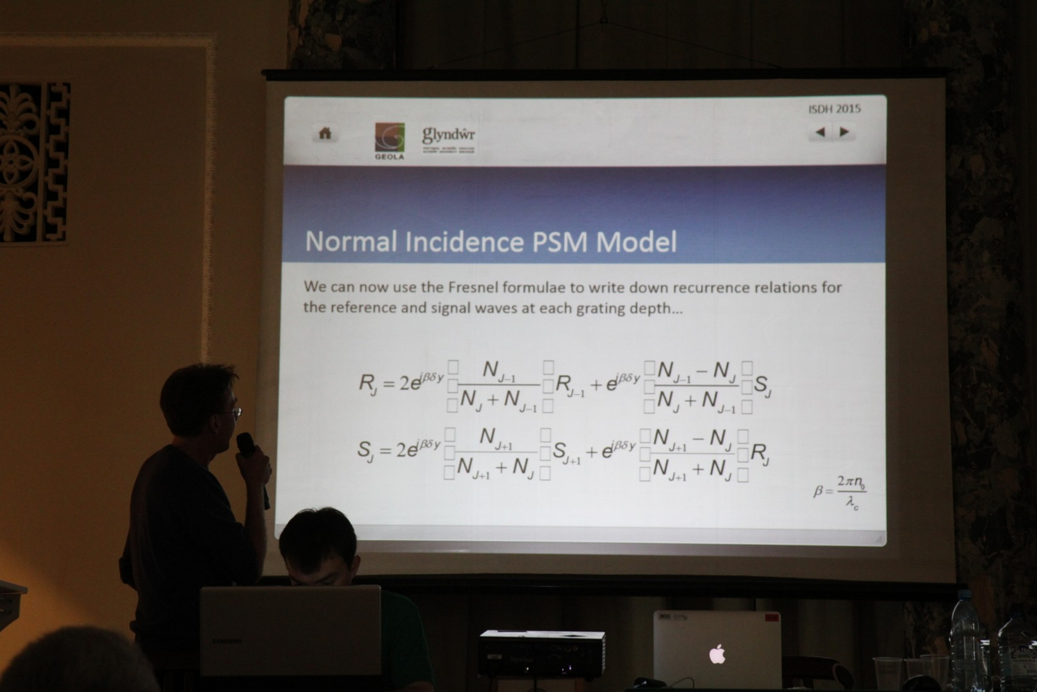
Normal Incidence PSM Model
We start by illuminating the grating with a plane wave
$R^{ext} = e^{i\beta y}$
$\beta = \frac{2\pi n_0}{\lambda_c}$
This is an exact solution of Maxwell's equations in the exterior region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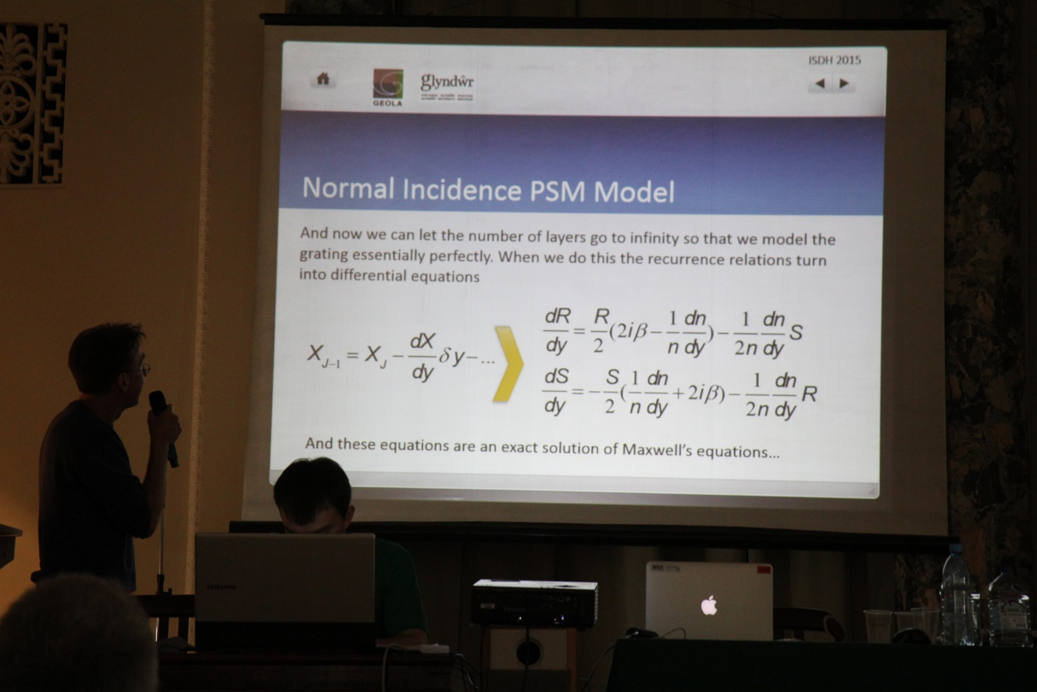
Normal Incidence PSM Model
We can now use the Fresnel formulae to write down recurrence relations for the reference and signal waves at each grating depth...
$R_J = 2e^{j\beta\delta y}\left(\frac{N_{J-1}}{N_J + N_{J-1}}\right)R_{J-1} + e^{j\beta\delta y}\left(\frac{N_{J-1} - N_J}{N_J + N_{J-1}}\right)S_J$
$S_J = 2e^{j\beta\delta y}\left(\frac{N_{J+1}}{N_{J+1} + N_J}\right)S_{J+1} + e^{j\beta\delta y}\left(\frac{N_{J+1} - N_J}{N_{J+1} + N_J}\right)R_J$
$\beta = \frac{2\pi n_0}{\lambda_c}$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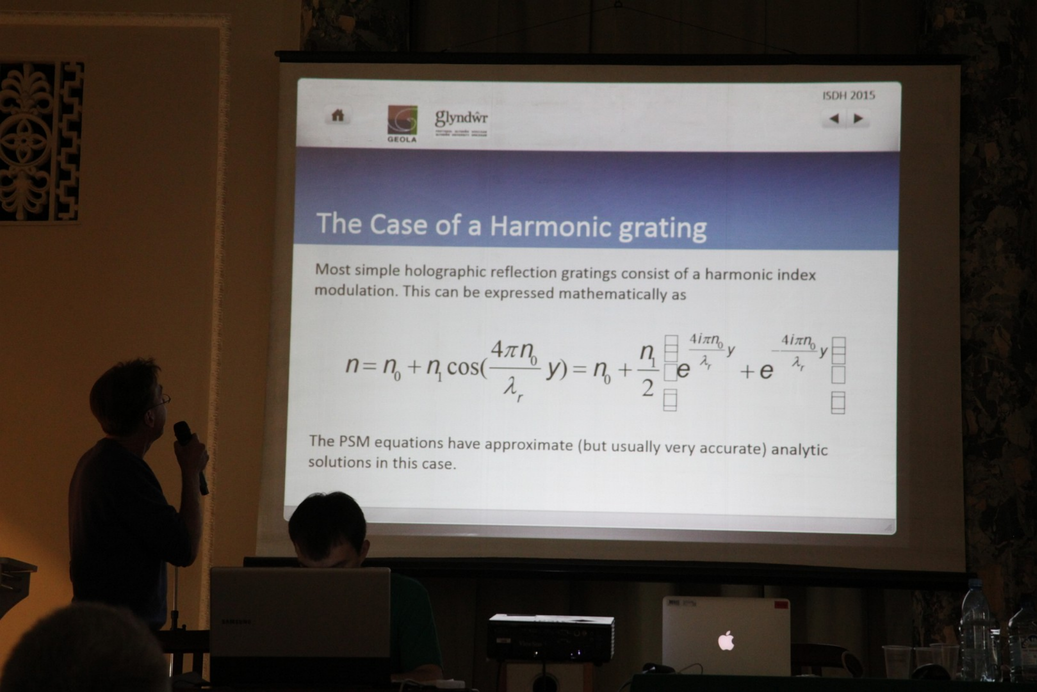
Normal Incidence PSM Model
And now we can let the number of layers go to infinity so that we model the grating essentially perfectly. When we do this the recurrence relations turn into differential equations
$X_{J-1} = X_J - \frac{dX}{dy}\delta y - \ldots$
$\frac{dR}{dy} = \frac{R}{2}(2i\beta - \frac{1}{n}\frac{dn}{dy}) - \frac{1}{2n}\frac{dn}{dy}S$
$\frac{dS}{dy} = -\frac{S}{2}(\frac{1}{n}\frac{dn}{dy} + 2i\beta) - \frac{1}{2n}\frac{dn}{dy}R$
And these equations are an exact solution of Maxwell's equations...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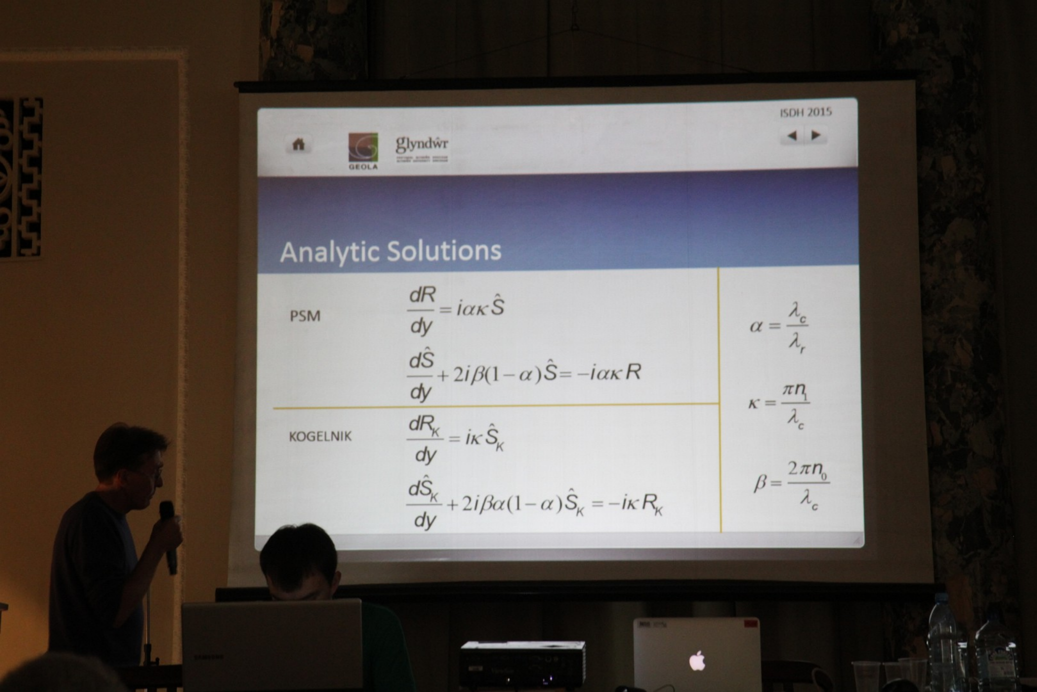
The Case of a Harmonic grating
Most simple holographic reflection gratings consist of a harmonic index modulation. This can be expressed mathematically as
$$n = n_0 + n_1 \cos\left(\frac{4\pi n_0}{\lambda_r} y\right) = n_0 + \frac{n_1}{2}\left(e^{\frac{4i\pi n_0}{\lambda_r} y} + e^{-\frac{4i\pi n_0}{\lambda_r} y} \right)$$
The PSM equations have approximate (but usually very accurate) analytic solutions in this case.



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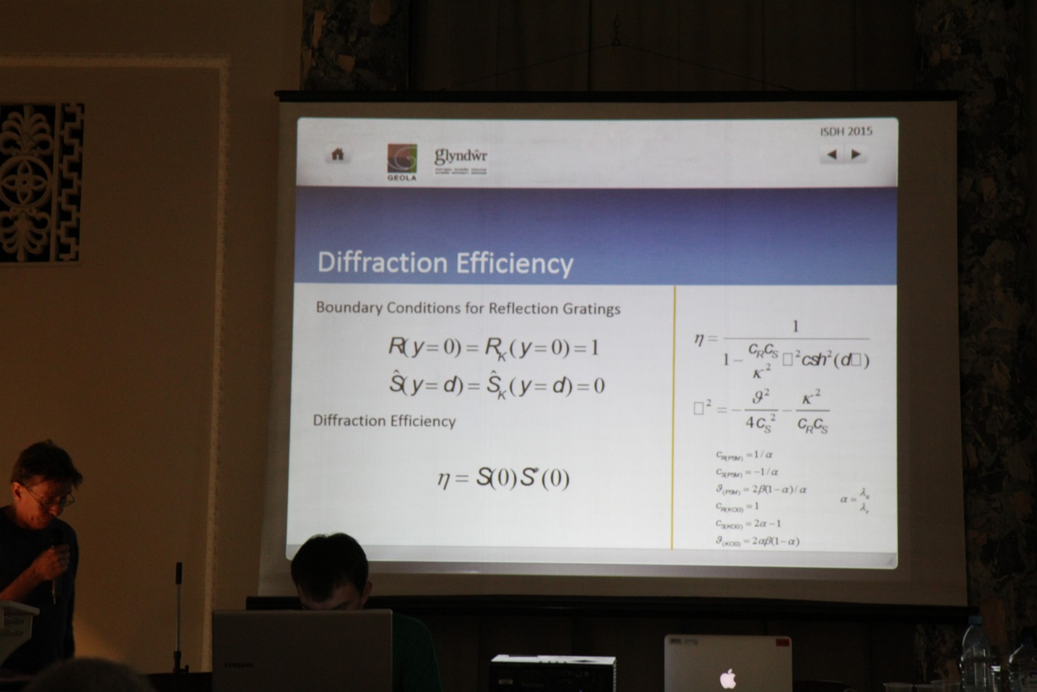
Analytic Solutions
PSM
$\frac{dR}{dy} = i\alpha\kappa\hat{S}$
$\frac{d\hat{S}}{dy} + 2i\beta(1-\alpha)\hat{S} = -i\alpha\kappa R$
KOGELNIK
$\frac{dR_K}{dy} = i\kappa\hat{S}_K$
$\frac{d\hat{S}_K}{dy} + 2i\beta\alpha(1-\alpha)\hat{S}_K = -i\kappa R_K$
$\alpha = \frac{\lambda_c}{\lambda_r}$
$\kappa = \frac{\pi n_1}{\lambda_c}$
$\beta = \frac{2\pi n_0}{\lambda_c}$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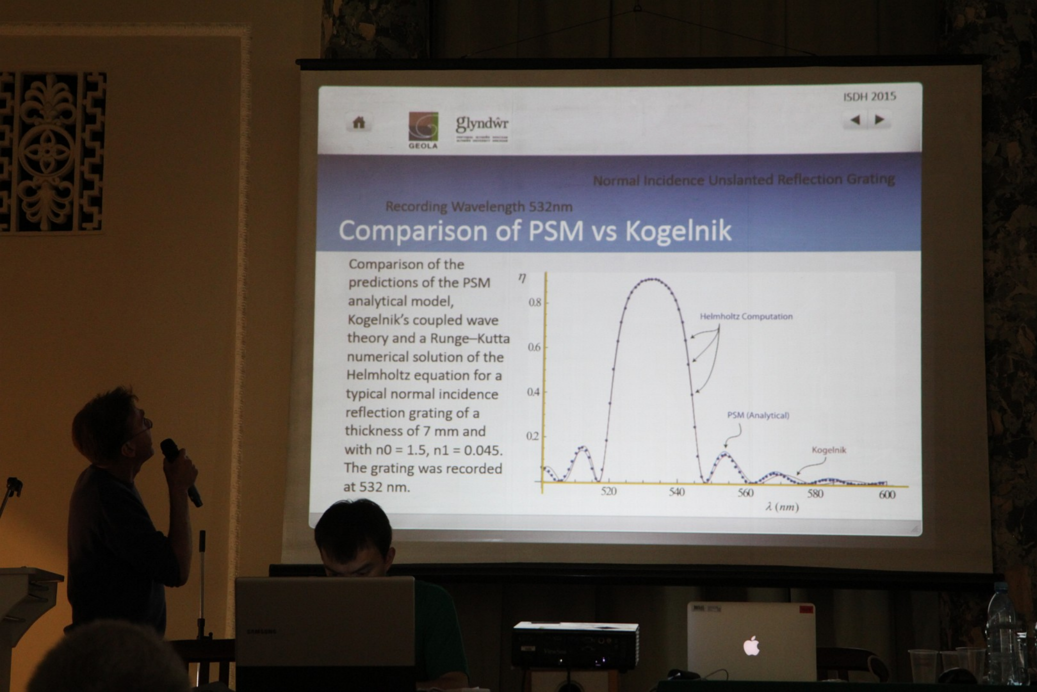
Diffraction Efficiency
Boundary Conditions for Reflection Gratings
$R(y=0) = R_K(y=0) = 1$
$\hat{S}(y=d) = \hat{S}_K(y=d) = 0$
Diffraction Efficiency
$\eta = S(0)S^*(0)$
$\eta = \frac{1}{1 - \frac{c_R c_S}{\kappa^2} \square^2 csh^2(d\square)}$
$\square^2 = -\frac{\vartheta^2}{4c_S^2} - \frac{\kappa^2}{c_R c_S}$
$c_{R(PSM)} = 1/\alpha$
$c_{S(PSM)} = -1/\alpha$
$\vartheta_{(PSM)} = 2\beta(1-\alpha)/\alpha$
$c_{R(KOG)} = 1$
$c_{S(KOG)} = 2\alpha - 1$
$\vartheta_{(KOG)} = 2\alpha\beta(1-\alpha)$
$\alpha = \frac{\lambda_c}{\lambda_r}$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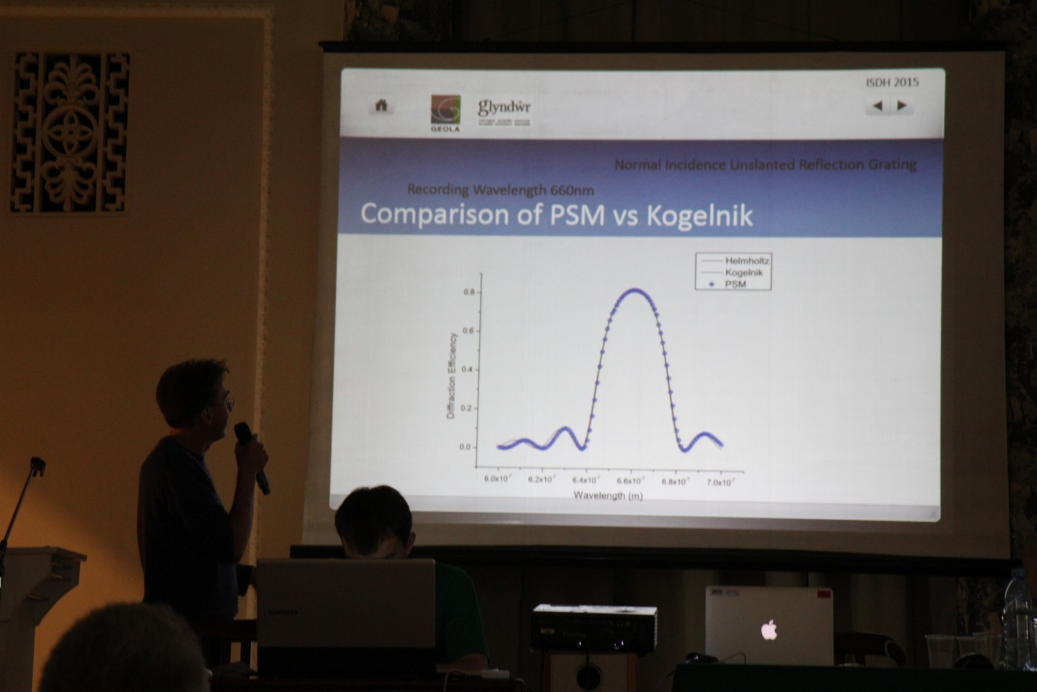
Normal Incidence Unslanted Reflection Grating
Recording Wavelength 532nm
Comparison of PSM vs Kogelnik
Comparison of the predictions of the PSM analytical model, Kogelnik's coupled wave theory and a Runge–Kutta numerical solution of the Helmholtz equation for a typical normal incidence reflection grating of a thickness of 7 mm and with n0 = 1.5, n1 = 0.045. The grating was recorded at 532 nm.
η
0.8
0.6
0.4
0.2
520
540
560
580
600
λ (nm)
Helmholtz Computation
PSM (Analytical)
Kogelnik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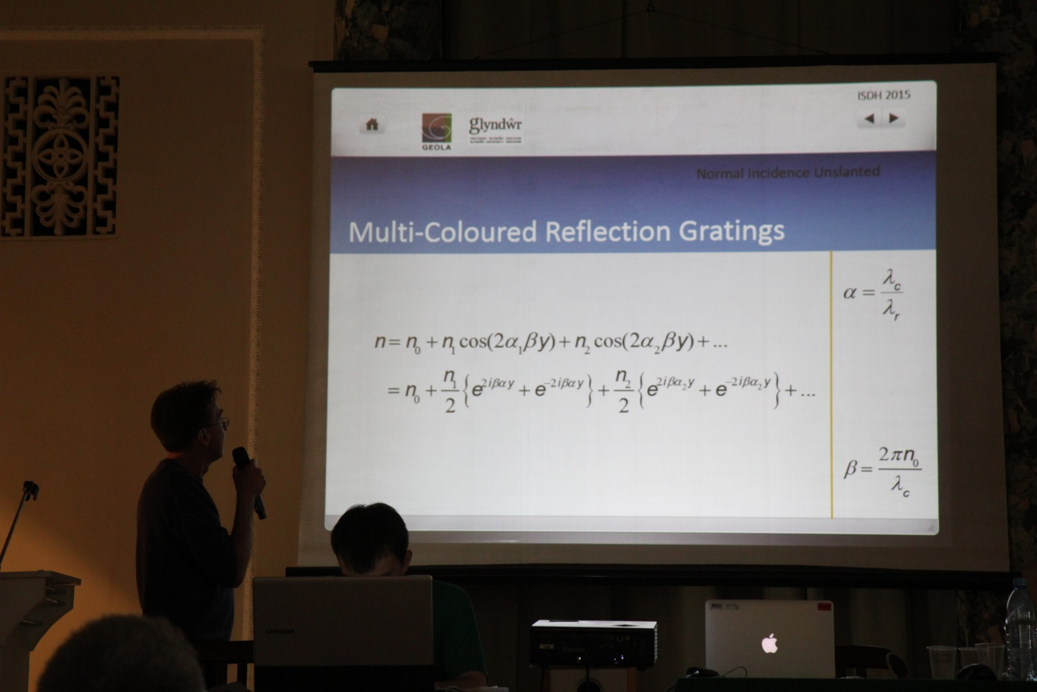
Normal Incidence Unslanted Reflection Grating
Recording Wavelength 660nm
Comparison of PSM vs Kogelnik
Helmholtz
Kogelnik
PSM
Diffraction Efficiency
Wavelength (m)
SAMSUNG



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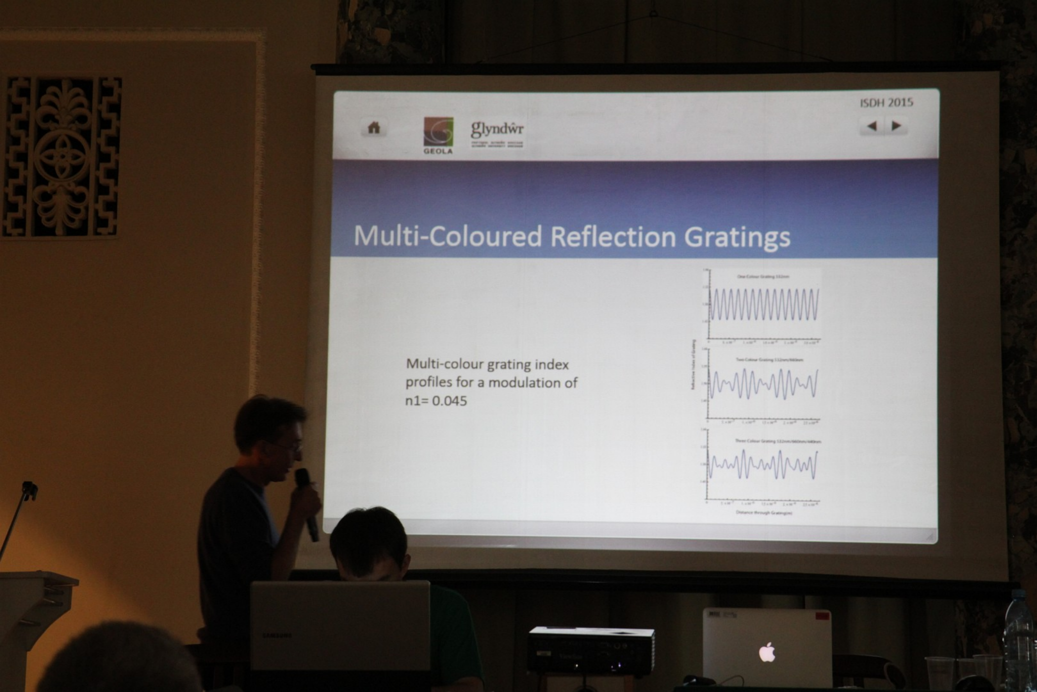
Normal Incidence Unslanted
Multi-Coloured Reflection Gratings
$n = n_0 + n_1 \cos(2\alpha_1 \beta y) + n_2 \cos(2\alpha_2 \beta y) + \dots$
$= n_0 + \frac{n_1}{2}\left\{e^{2i\beta\alpha y} + e^{-2i\beta\alpha y}\right\} + \frac{n_2}{2}\left\{e^{2i\beta\alpha_2 y} + e^{-2i\beta\alpha_2 y}\right\} + \dots$
$\alpha = \frac{\lambda_c}{\lambda_r}$
$\beta = \frac{2\pi n_0}{\lambda_c}$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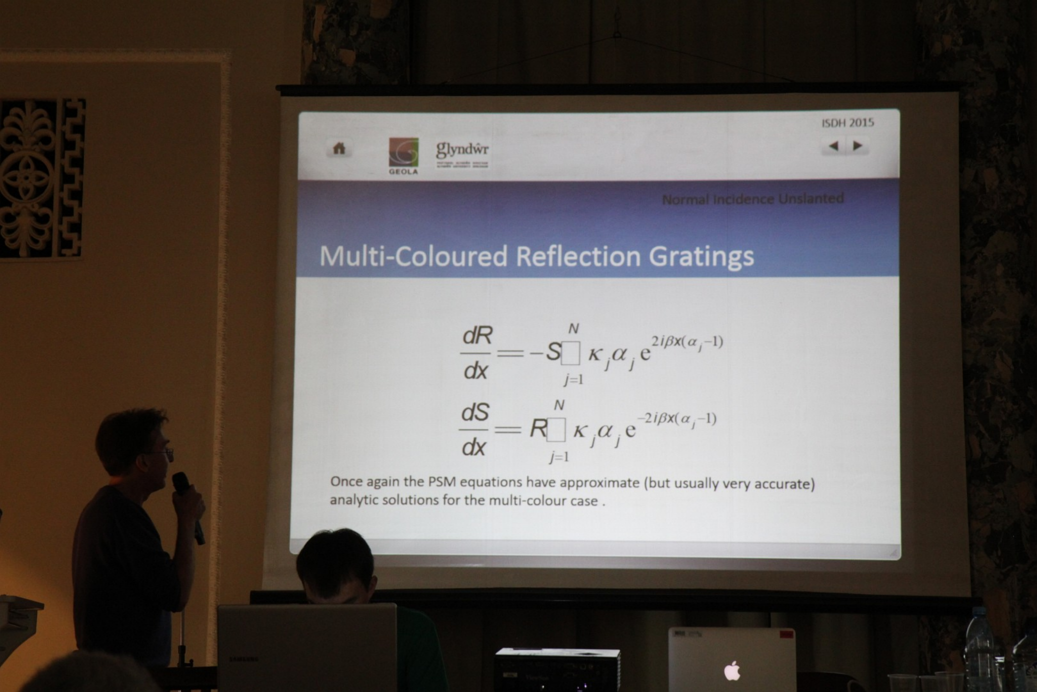
Multi-Coloured Reflection Gratings
Multi-colour grating index profiles for a modulation of n1= 0.045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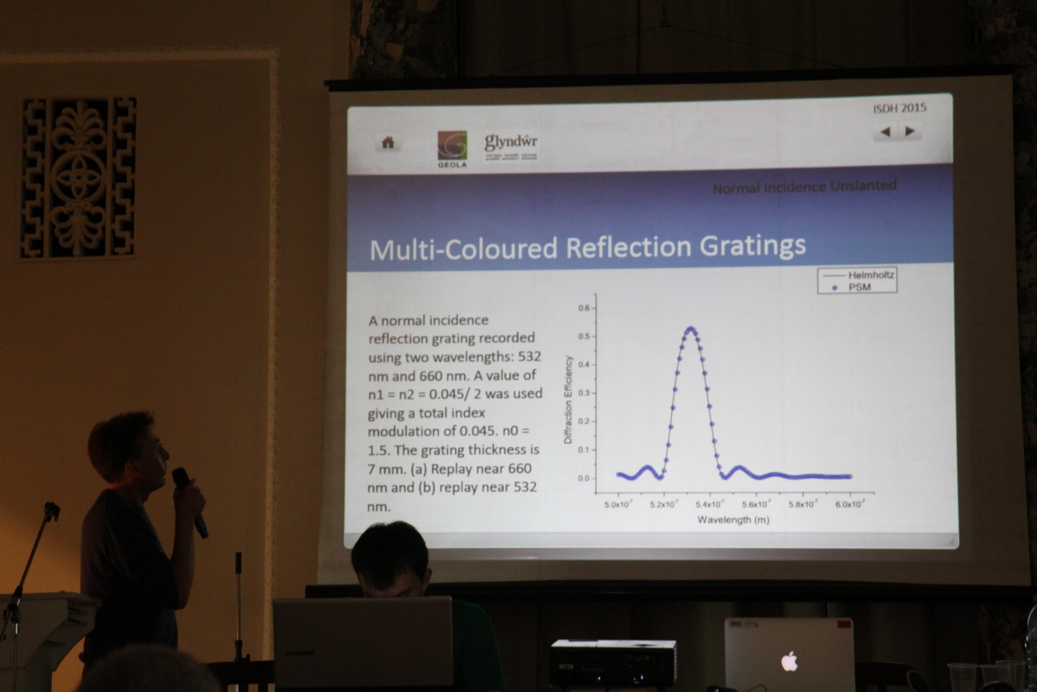
Normal Incidence Unslanted
Multi-Coloured Reflection Gratings
$$\frac{dR}{dx} = -S\sum_{j=1}^{N} \kappa_j \alpha_j e^{2i\beta x(\alpha_j - 1)}$$
$$\frac{dS}{dx} = R\sum_{j=1}^{N} \kappa_j \alpha_j e^{-2i\beta x(\alpha_j - 1)}$$
Once again the PSM equations have approximate (but usually very accurate) analytic solutions for the multi-colour case .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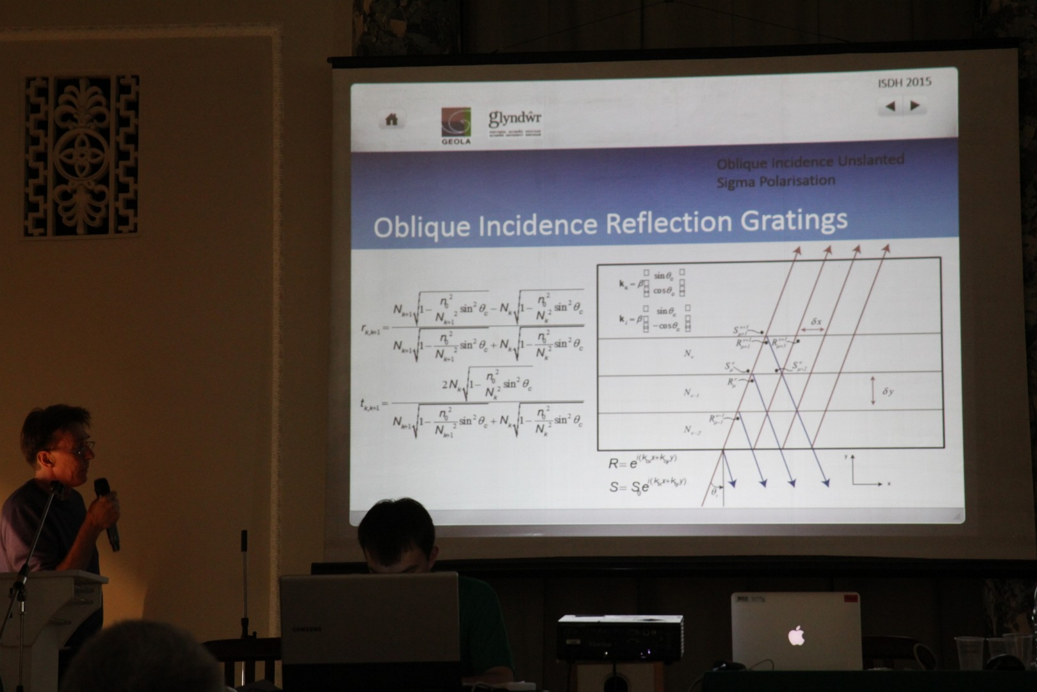
Normal Incidence Unslanted
Multi-Coloured Reflection Gratings
A normal incidence reflection grating recorded using two wavelengths: 532 nm and 660 nm. A value of n1 = n2 = 0.045/ 2 was used giving a total index modulation of 0.045. n0 = 1.5. The grating thickness is 7 mm. (a) Replay near 660 nm and (b) replay near 532 nm.
Helmholtz
PSM
Diffraction Efficiency
0.6
0.5
0.4
0.3
0.2
0.1
0.0
5.0x10⁻⁷
5.2x10⁻⁷
5.4x10⁻⁷
5.6x10⁻⁷
5.8x10⁻⁷
6.0x10⁻⁷
Wavelength (m)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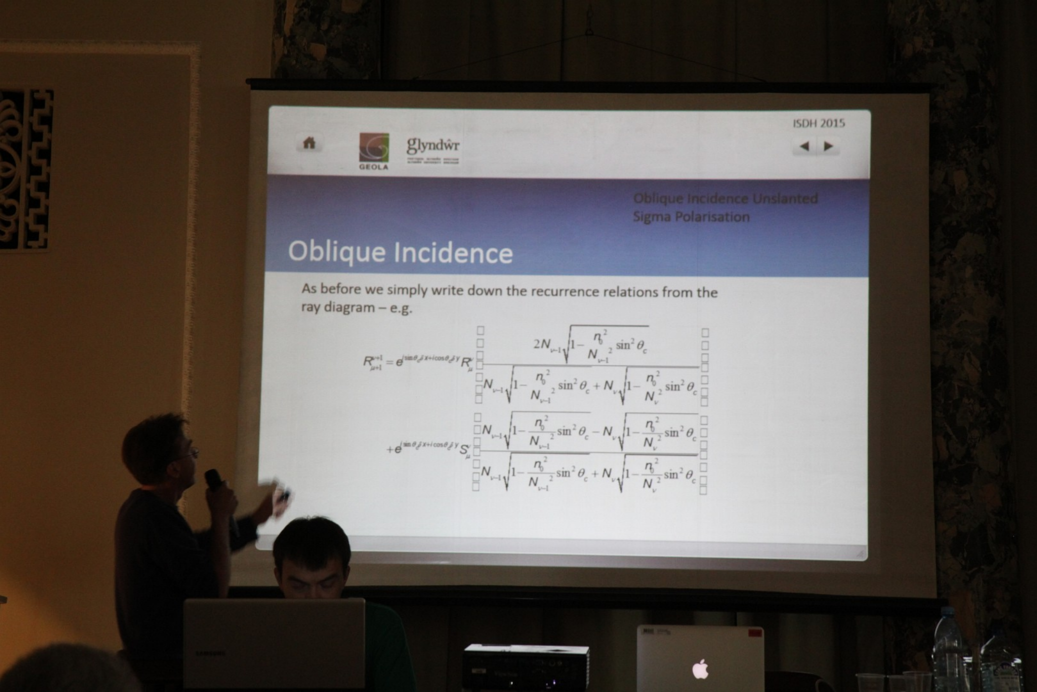
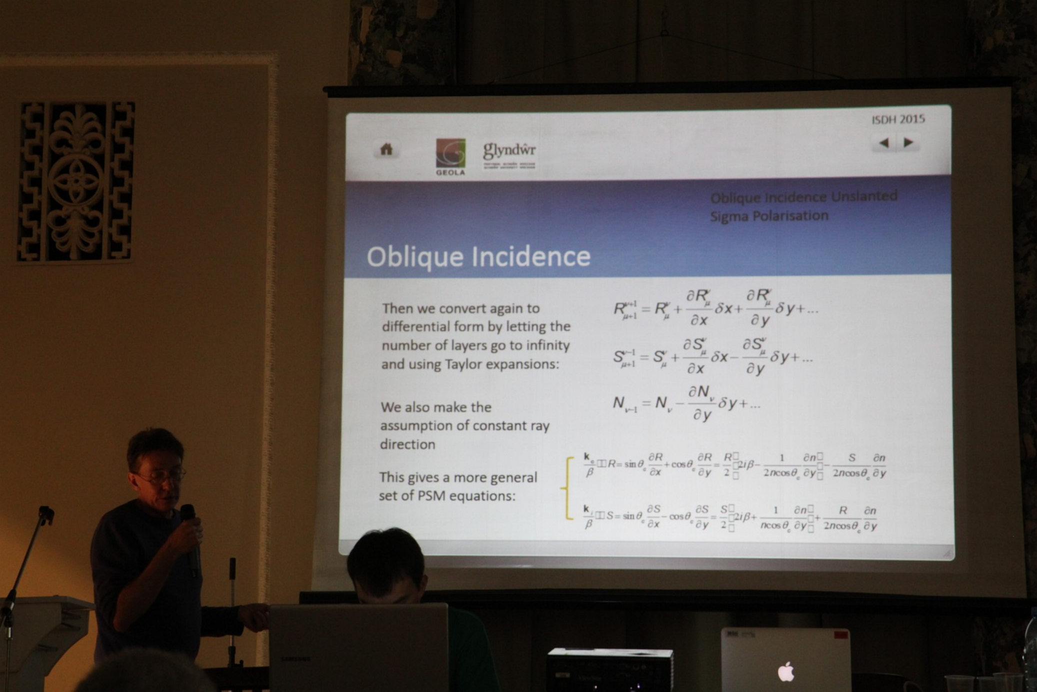
Oblique incidence Unslanted
Sigma Polarisation
Oblique Incidence Reflection Gratings
δx
δy


ISDH 2015
Glyndŵr
GEOLA
Oblique Incidence Unslanted
Sigma Polarisation
Oblique Incidence
As before we simply write down the recurrence relations from the ray diagram – e.g.
$$R_{\mu+1}^{\nu+1} = e^{i\sin\theta_c\beta x + i\cos\theta_c\beta y} R_{\mu}^{\nu} \frac{2N_{\nu-1}\sqrt{1-\frac{n_0^2}{N_{\nu-1}^2}\sin^2\theta_c}}{N_{\nu-1}\sqrt{1-\frac{n_0^2}{N_{\nu-1}^2}\sin^2\theta_c}+N_{\nu}\sqrt{1-\frac{n_0^2}{N_{\nu}^2}\sin^2\theta_c}} + e^{i\sin\theta_c\beta x + i\cos\theta_c\beta y} S_{\mu}^{\nu} \frac{N_{\nu-1}\sqrt{1-\frac{n_0^2}{N_{\nu-1}^2}\sin^2\theta_c}-N_{\nu}\sqrt{1-\frac{n_0^2}{N_{\nu}^2}\sin^2\theta_c}}{N_{\nu-1}\sqrt{1-\frac{n_0^2}{N_{\nu-1}^2}\sin^2\theta_c}+N_{\nu}\sqrt{1-\frac{n_0^2}{N_{\nu}^2}\sin^2\theta_c}}$$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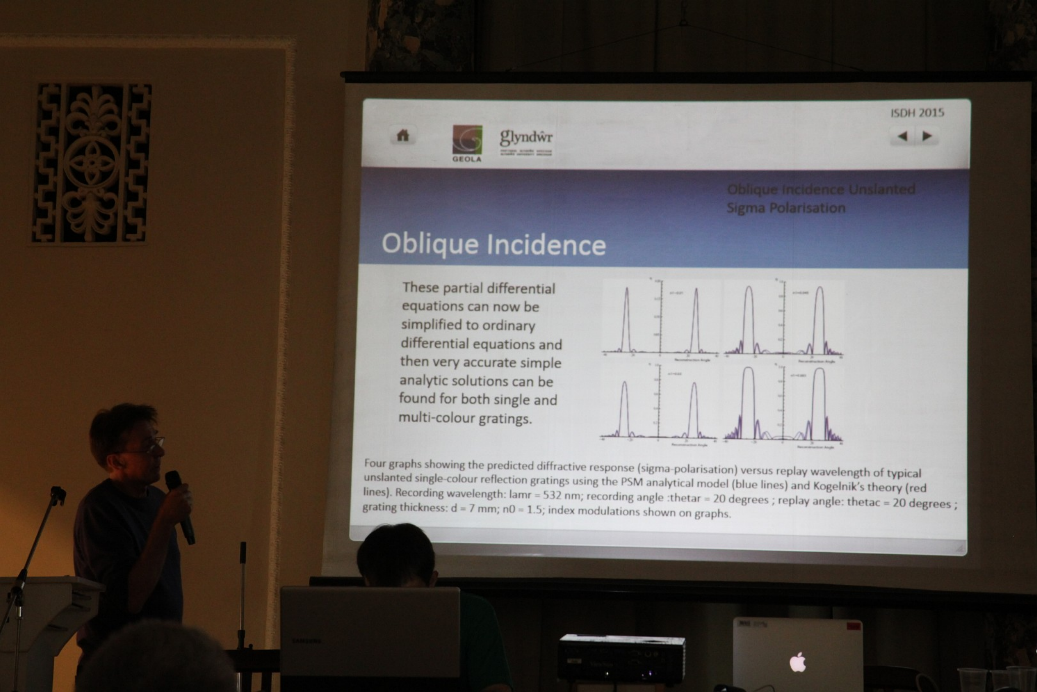
Oblique Incidence Unslanted
Sigma Polarisation
Oblique Incidence
Then we convert again to differential form by letting the number of layers go to infinity and using Taylor expansions:
$R_{\mu+1}^{\nu+1} = R_{\mu}^{\nu} + \frac{\partial R_{\mu}^{\nu}}{\partial x}\delta x + \frac{\partial R_{\mu}^{\nu}}{\partial y}\delta y + \ldots$
$S_{\mu+1}^{\nu-1} = S_{\mu}^{\nu} + \frac{\partial S_{\mu}^{\nu}}{\partial x}\delta x - \frac{\partial S_{\mu}^{\nu}}{\partial y}\delta y + \ldots$
We also make the assumption of constant ray direction
$N_{\nu-1} = N_{\nu} - \frac{\partial N_{\nu}}{\partial y}\delta y + \ldots$
This gives a more general set of PSM equations:
$\frac{\mathbf{k}_c}{\beta} \square\square R = \sin\theta_c \frac{\partial R}{\partial x} + \cos\theta_c \frac{\partial R}{\partial y} = \frac{R}{2}\square 2i\beta - \frac{1}{2n\cos\theta_c}\frac{\partial n}{\partial y}\square - \frac{S}{2n\cos\theta_c}\frac{\partial n}{\partial y}$
$\frac{\mathbf{k}_i}{\beta} \square\square S = \sin\theta_c \frac{\partial S}{\partial x} - \cos\theta_c \frac{\partial S}{\partial y} = \frac{S}{2}\square 2i\beta + \frac{1}{n\cos\theta_c}\frac{\partial n}{\partial y}\square + \frac{R}{2n\cos\theta_c}\frac{\partial n}{\partial y}$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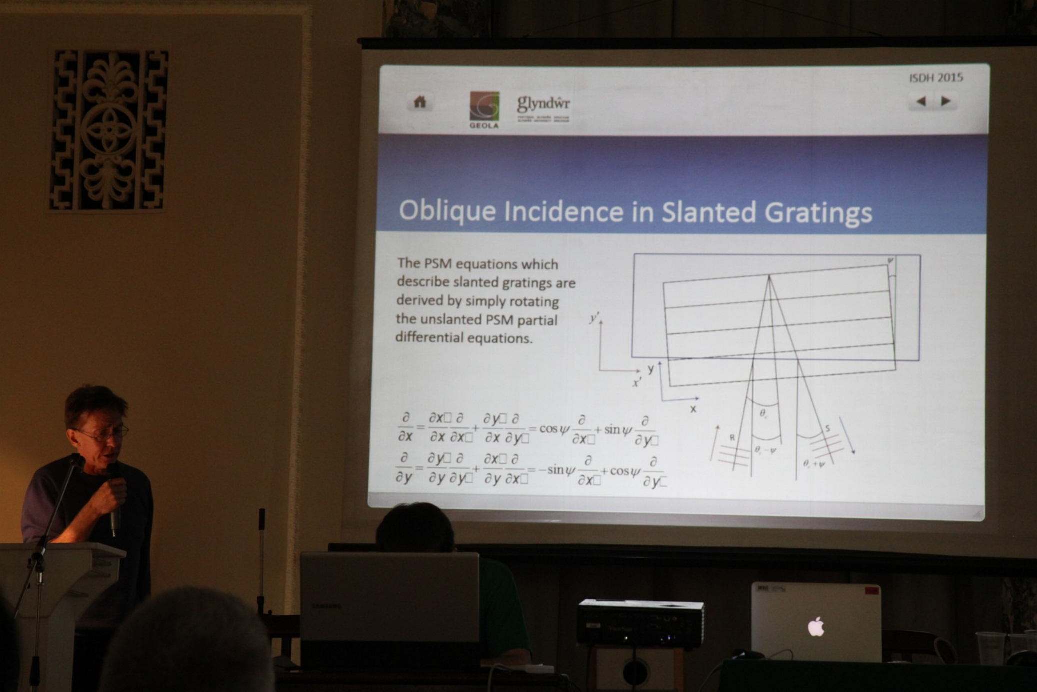
Oblique Incidence Unslanted
Sigma Polarisation
Oblique Incidence
These partial differential equations can now be simplified to ordinary differential equations and then very accurate simple analytic solutions can be found for both single and multi-colour gratings.
Four graphs showing the predicted diffractive response (sigma-polarisation) versus replay wavelength of typical unslanted single-colour reflection gratings using the PSM analytical model (blue lines) and Kogelnik's theory (red lines). Recording wavelength: lamr = 532 nm; recording angle :thetar = 20 degrees ; replay angle: thetac = 20 degrees ; grating thickness: d = 7 mm; n0 = 1.5; index modulations shown on graphs.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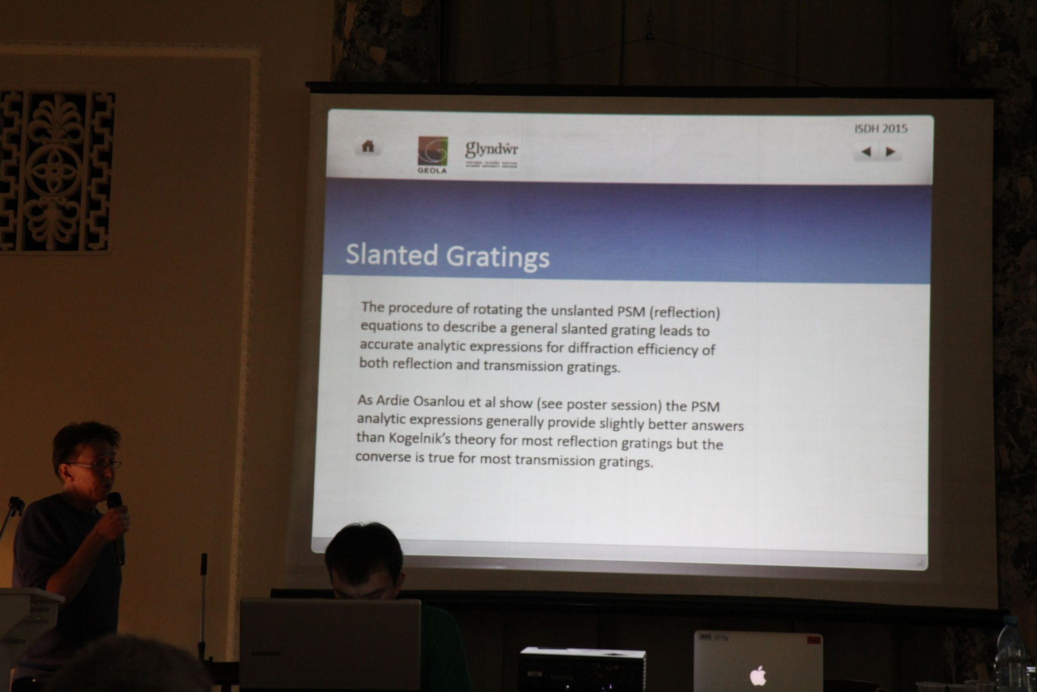
Oblique Incidence in Slanted Gratings
The PSM equations which describe slanted gratings are derived by simply rotating the unslanted PSM partial differential equations.
$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos\psi \frac{\partial}{\partial x'} + \sin\psi \frac{\partial}{\partial y'}$
$\frac{\partial}{\partial y} = \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} + \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} = -\sin\psi \frac{\partial}{\partial x'} + \cos\psi \frac{\partial}{\partial y'}$
y'
x'
y
x
R
S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Slanted Gratings
The procedure of rotating the unslanted PSM (reflection) equations to describe a general slanted grating leads to accurate analytic expressions for diffraction efficiency of both reflection and transmission gratings.
As Ardie Osanlou et al show (see poster session) the PSM analytic expressions generally provide slightly better answers than Kogelnik's theory for most reflection gratings but the converse is true for most transmission gratings.


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D. Brotherton-Ratcliffe, "Analytical treatment of the polychromatic spatially multiplexed volume holographic grating," Appl. Opt. 51, 7188-7199 (2012).
Multiplexed Reflection Gratings
The PSM theory may be extended to describe spatially multiplexed (and multi-colour) gratings and gives analytical results very similar to N-Coupled Wave theory.
(a)
(b)
(c)
Fig. 3. Example of a spatially multiplexed phase reflection grating. The grating, whose (x, y) index distribution is shown in (c), is formed by the sequential recording of the two simple gratings shown in (a) and (b). Each diagram shows a section of size 0.5 μm by 2 μm. Each simple grating has been recorded with a reference beam angle of Φ_r = 30° and with a wavelength of 532 nm. One grating has a slope of ψ_1 = 20° and the other has a slope of ψ_2 = −20°. Note that the form of the multiplexed grating in (c) is fundamentally different from the characteristic linear shape of its component simple gratings of (a) and (b). Note also that identical index modulations for each of the two component gratings have been assumed in (c).
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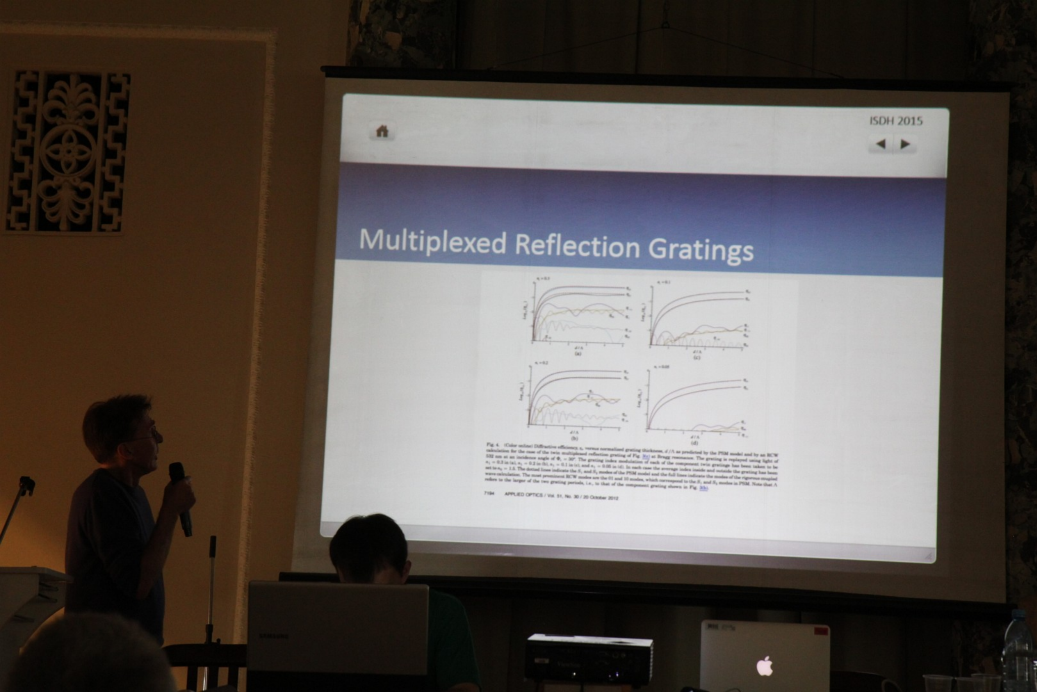
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Multiplexed Reflection Gratings
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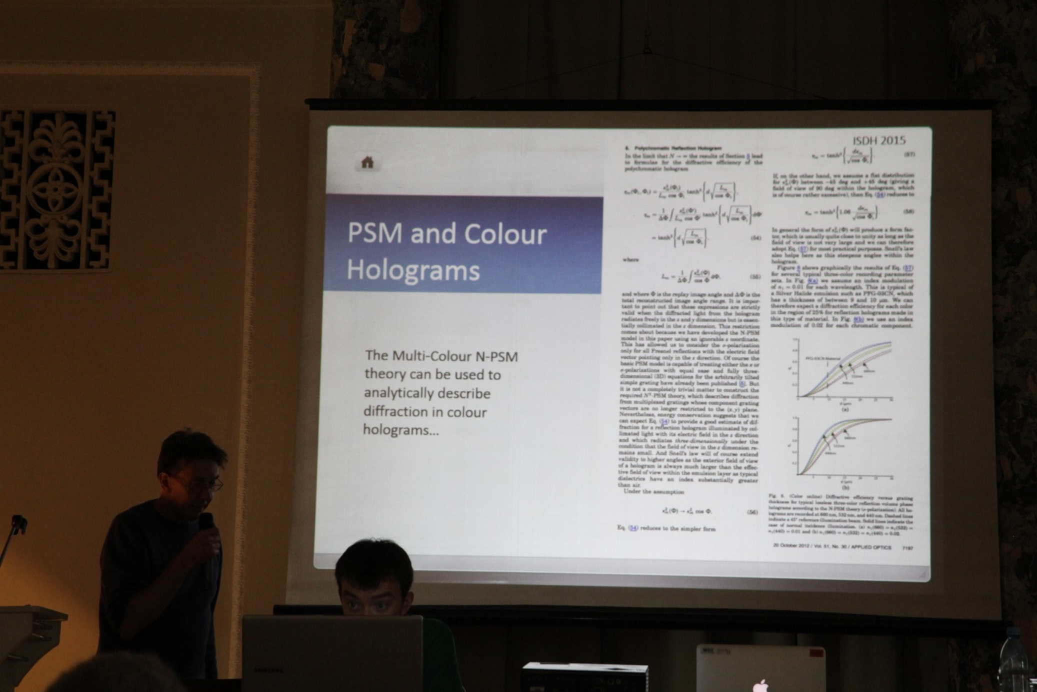
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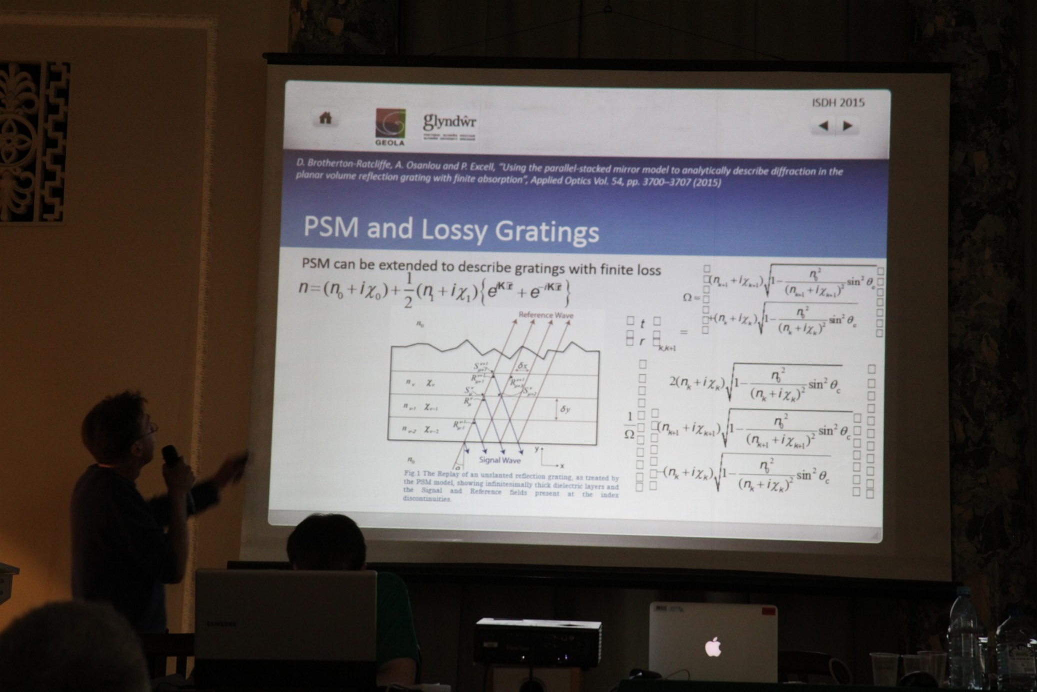
PSM and Colour Holograms
The Multi-Colour N-PSM theory can be used to analytically describe diffraction in colour holograms...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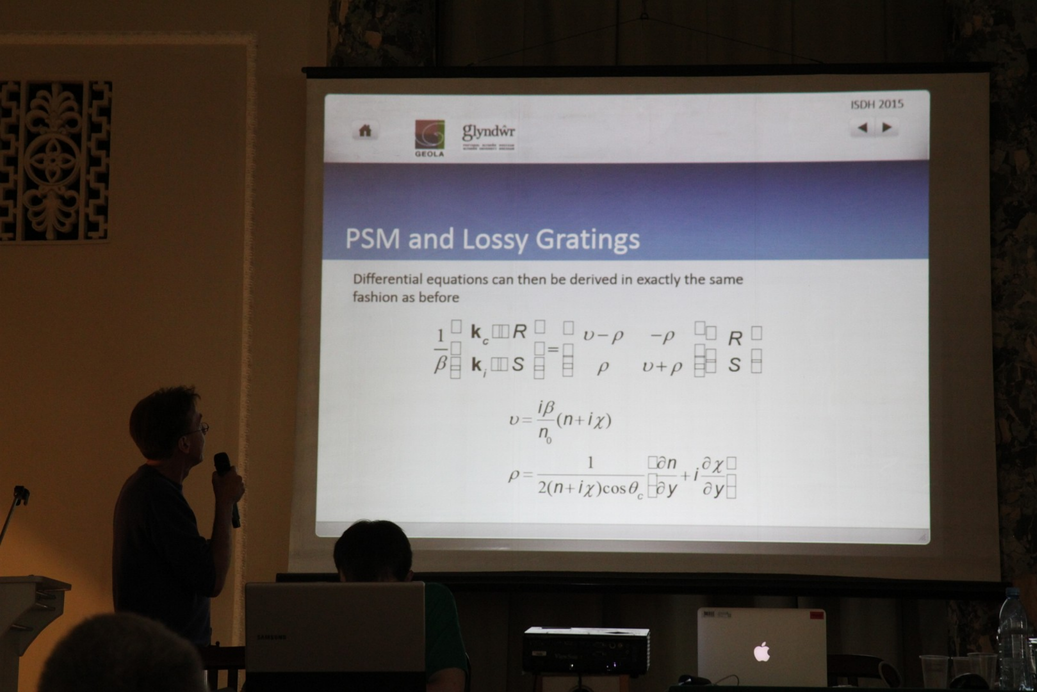
D. Brotherton-Ratcliffe, A. Osanlou and P. Excell, "Using the parallel-stacked mirror model to analytically describe diffraction in the planar volume reflection grating with finite absorption", Applied Optics Vol. 54, pp. 3700–3707 (2015)
PSM and Lossy Gratings
PSM can be extended to describe gratings with finite loss
$n=(n_0+i\chi_0)+\frac{1}{2}(n_1+i\chi_1)\left\{e^{i\mathbf{K}\cdot\mathbf{r}}+e^{-i\mathbf{K}\cdot\mathbf{r}}\right\}$
Reference Wave
Signal Wave
δy
Fig.1 The Replay of an unslanted reflection grating, as treated by the PSM model, showing infinitesimally thick dielectric layers and the Signal and Reference fields present at the index discontinuities.
$\Omega=(n_{k+1}+i\chi_{k+1})\sqrt{1-\frac{n_0^2}{(n_{k+1}+i\chi_{k+1})^2}\sin^2\theta_c}+(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}$
$2(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}$
$(n_{k+1}+i\chi_{k+1})\sqrt{1-\frac{n_0^2}{(n_{k+1}+i\chi_{k+1})^2}\sin^2\theta_c}$
$-(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}$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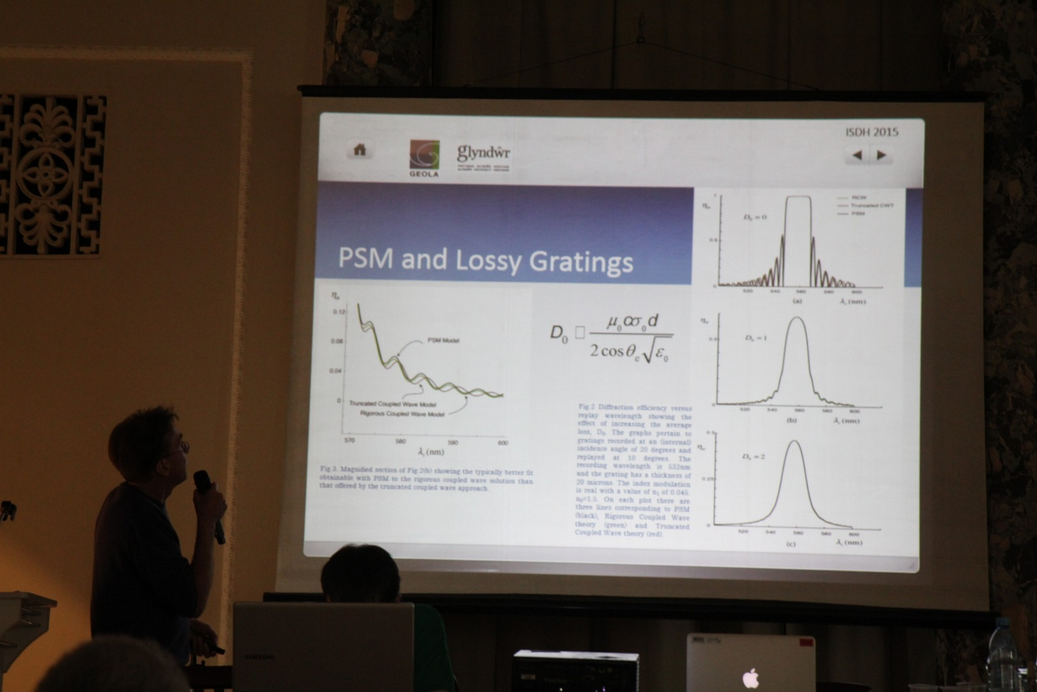
PSM and Lossy Gratings
Differential equations can then be derived in exactly the same fashion as before
$\frac{1}{\beta}\begin{pmatrix} \mathbf{k}_c \cdot \nabla R \\ \mathbf{k}_i \cdot \nabla S \end{pmatrix} = \begin{pmatrix} \upsilon-\rho & -\rho \\ \rho & \upsilon+\rho \end{pmatrix}\begin{pmatrix} R \\ S \end{pmatrix}$
$\upsilon = \frac{i\beta}{n_0}(n+i\chi)$
$\rho = \frac{1}{2(n+i\chi)\cos\theta_c}\left(\frac{\partial n}{\partial y}+i\frac{\partial \chi}{\partial y}\right)$



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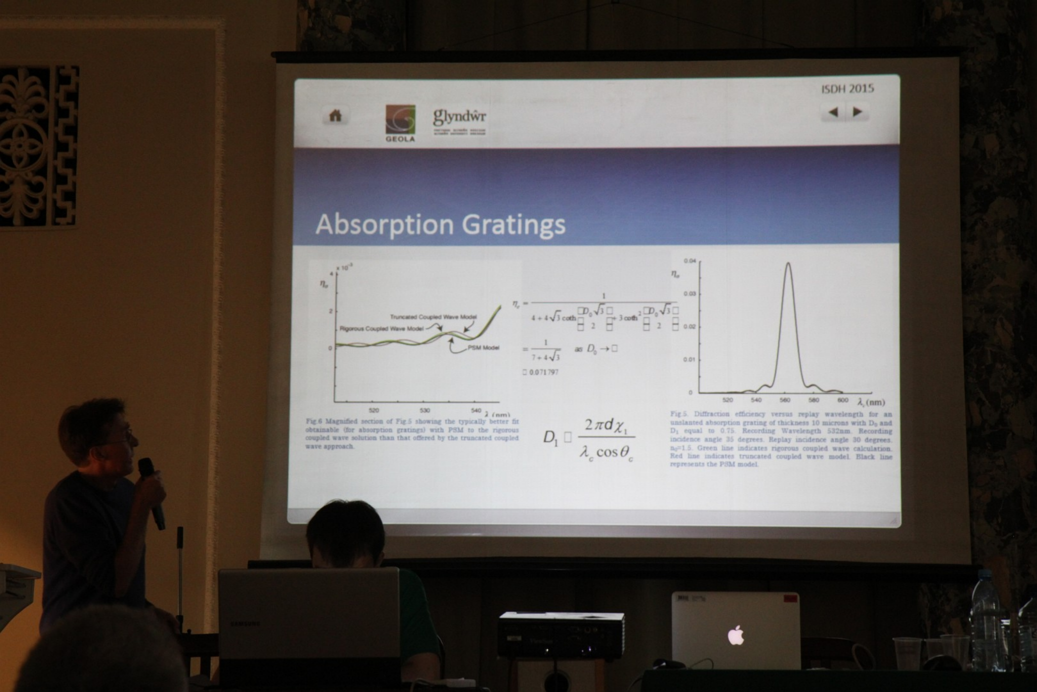
PSM and Lossy Gratings
PSM Model
Truncated Coupled Wave Model
Rigorous Coupled Wave Model
λ (nm)
Fig.3. Magnified section of Fig.2(b) showing the typically better fit obtainable with PSM to the rigorous coupled wave solution than that offered by the truncated coupled wave approach.
$D_0 \square \frac{\mu_0 \alpha \sigma_0 d}{2\cos\theta_c\sqrt{\varepsilon_0}}$
Fig.2 Diffraction efficiency versus replay wavelength showing the effect of increasing the average loss, D0. The graphs pertain to gratings recorded at an (internal) incidence angle of 20 degrees and replayed at 10 degrees. The recording wavelength is 532nm and the grating has a thickness of 20 microns. The index modulation is real with a value of n1 of 0.045. n0=1.5. On each plot there are three lines corresponding to PSM (black), Rigorous Coupled Wave theory (green) and Truncated Coupled Wave theory (red).
(a)
(b)
(c)



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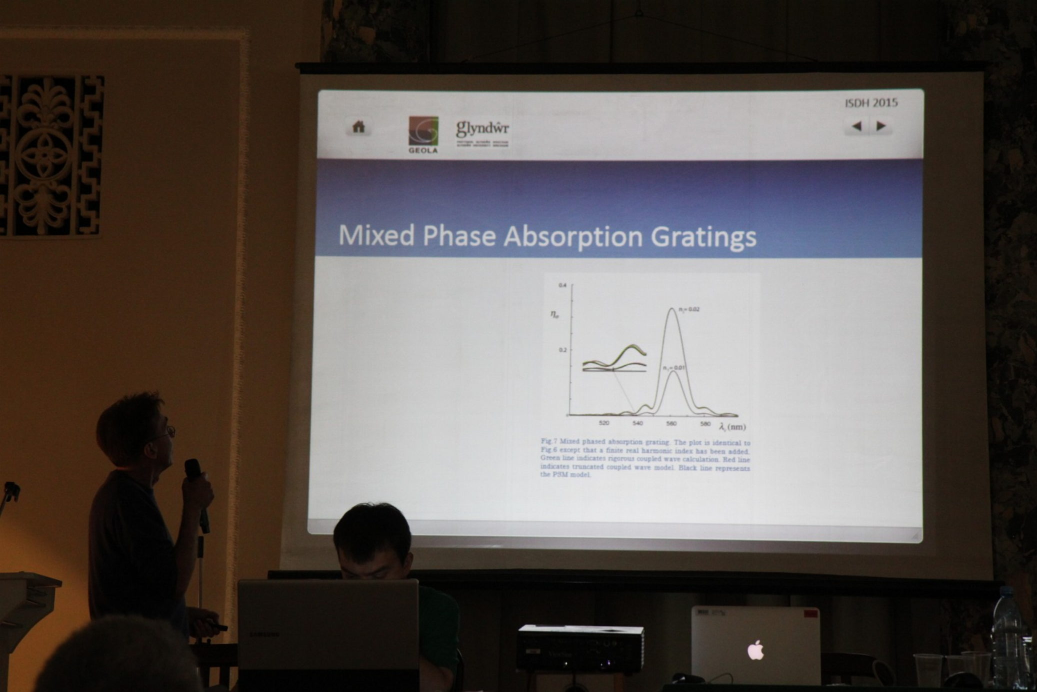
Absorption Gratings
Truncated Coupled Wave Model
Rigorous Coupled Wave Model
PSM Model
Fig.6 Magnified section of Fig.5 showing the typically better fit obtainable (for absorption gratings) with PSM to the rigorous coupled wave solution than that offered by the truncated coupled wave approach.
$\eta_e = \frac{1}{7+4\sqrt{3}}$ as $D_0 \to$
0.071797
$D_1 = \frac{2\pi d \chi_1}{\lambda_c \cos\theta_c}$
λc (nm)
Fig.5. Diffraction efficiency versus replay wavelength for an unslanted absorption grating of thickness 10 microns with D_0 and D_1 equal to 0.75. Recording Wavelength 532nm. Recording incidence angle 35 degrees. Replay incidence angle 30 degrees. n_0=1.5. Green line indicates rigorous coupled wave calculation. Red line indicates truncated coupled wave model. Black line represents the PSM model.



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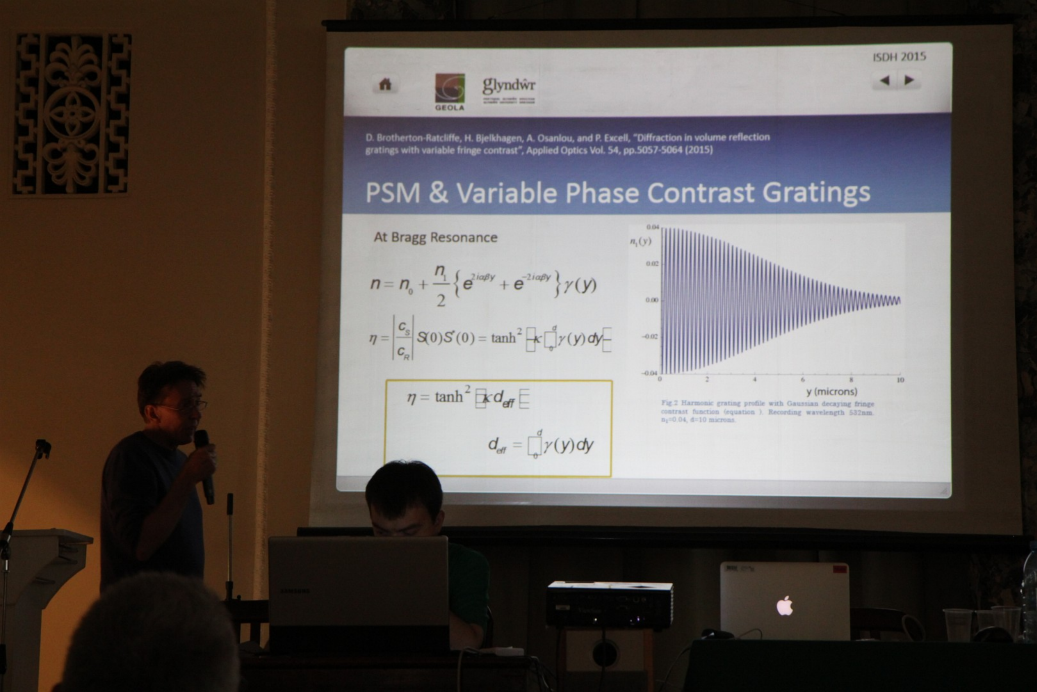
Mixed Phase Absorption Gratings
0.4
0.2
η_σ
$n_1 = 0.02$
$n_1 = 0.01$
520
540
560
580
λ_c (nm)
Fig.7 Mixed phased absorption grating. The plot is identical to Fig.6 except that a finite real harmonic index has been added. Green line indicates rigorous coupled wave calculation. Red line indicates truncated coupled wave model. Black line represents the PSM model.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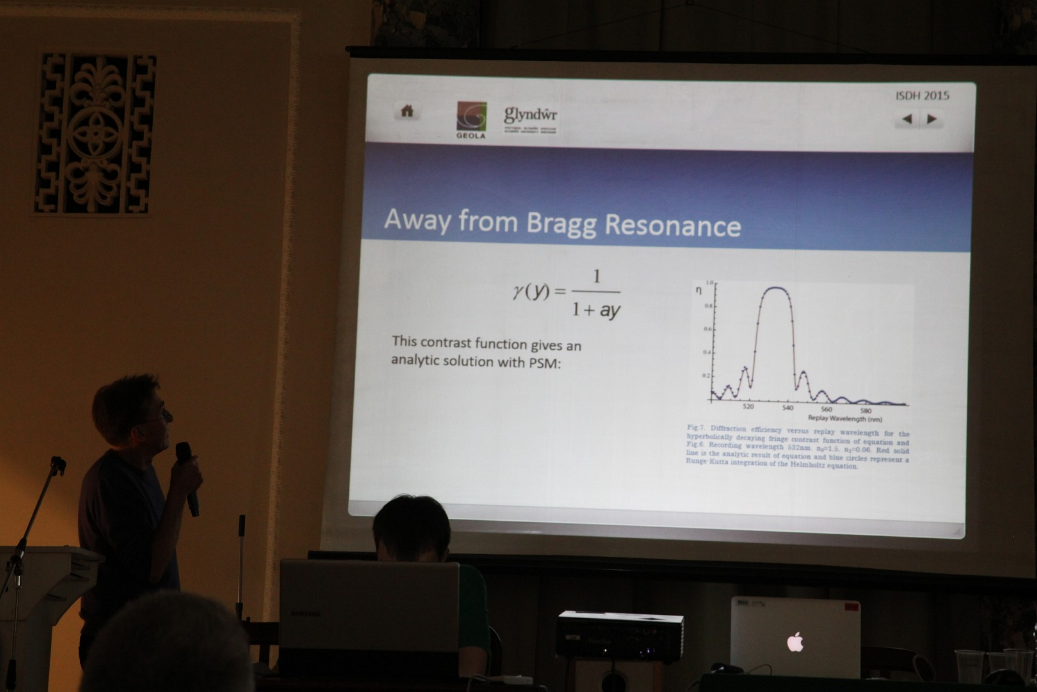
D. Brotherton-Ratcliffe, H. Bjelkhagen, A. Osanlou, and P. Excell, "Diffraction in volume reflection gratings with variable fringe contrast", Applied Optics Vol. 54, pp.5057-5064 (2015)
PSM & Variable Phase Contrast Gratings
At Bragg Resonance
$n = n_0 + \frac{n_1}{2}\left\{e^{2i\alpha\beta y} + e^{-2i\alpha\beta y}\right\}\gamma(y)$
$\eta = \left|\frac{c_S}{c_R}\right| S(0)S^*(0) = \tanh^2 \left(\kappa \int_0^d \gamma(y)\,dy\right)$
$\eta = \tanh^2 \left(\kappa d_{eff}\right)$
$d_{eff} = \int_0^d \gamma(y)\,dy$
$n_1(y)$
0.04
0.02
0.00
-0.02
-0.04
0
2
4
6
8
10
y (microns)
Fig.2 Harmonic grating profile with Gaussian decaying fringe contrast function (equation). Recording wavelength 532nm. n_1=0.04, d=10 microns.



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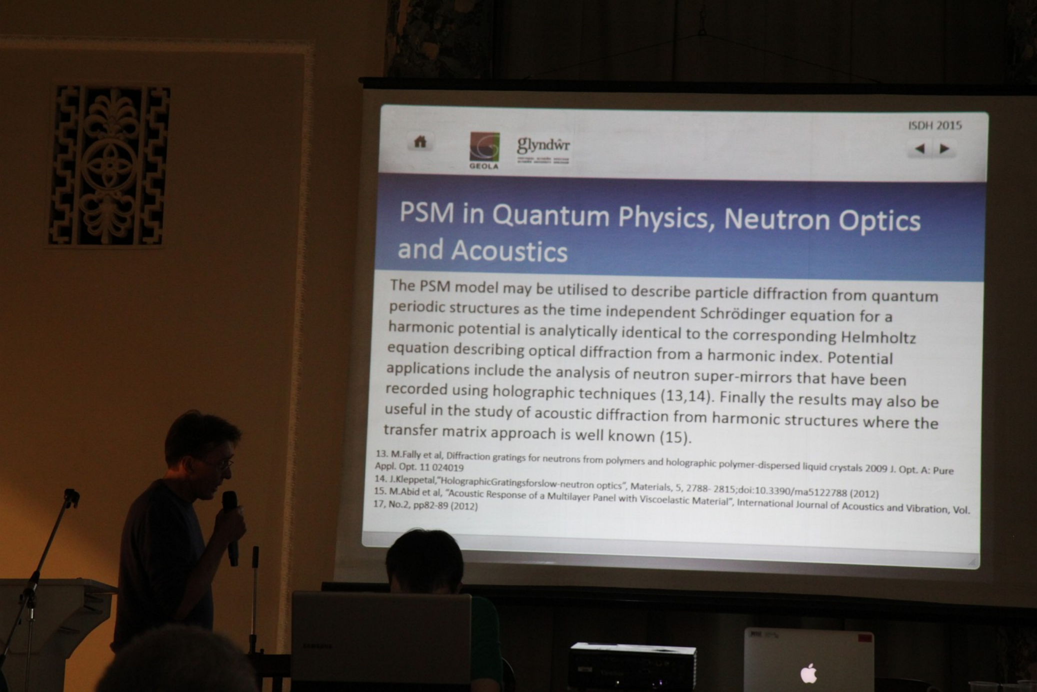
Away from Bragg Resonance
γ(y) = 1 / (1 + ay)
This contrast function gives an
analytic solution with PSM:
η
520
540
560
580
Replay Wavelength (nm)
Fig.7. Diffraction efficiency versus replay wavelength for the hyperbolically decaying fringe contrast function of equation and Fig.6. Recording wavelength 532nm. n0=1.5. n1=0.06. Red solid line is the analytic result of equation and blue circles represent a Runge-Kutta integration of the Helmholtz equation.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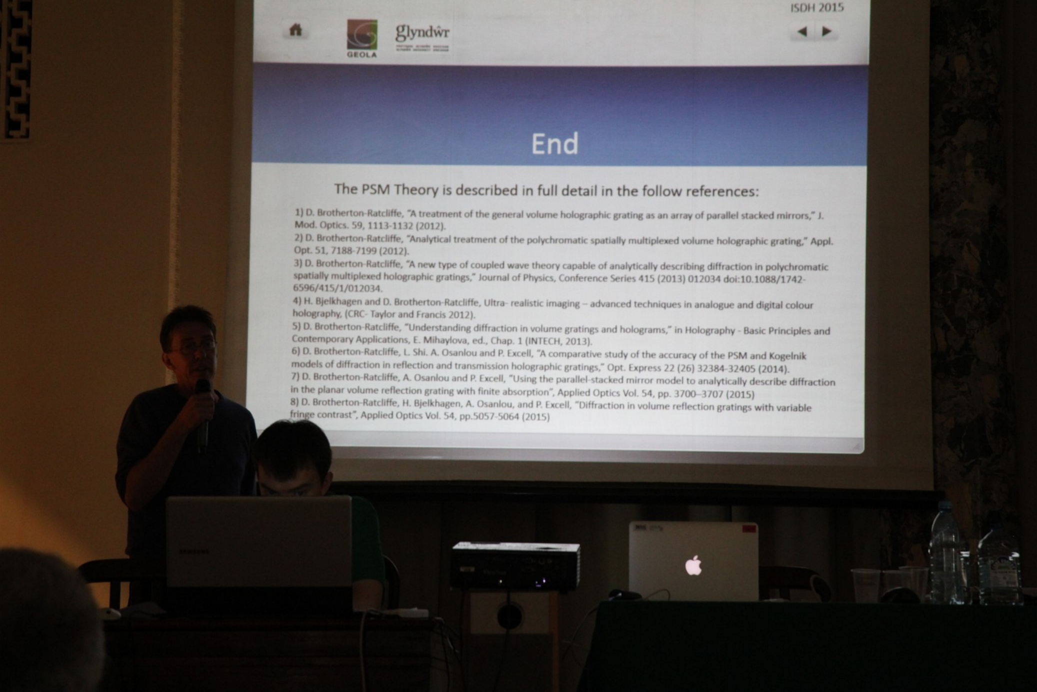
PSM in Quantum Physics, Neutron Optics and Acoustics
The PSM model may be utilised to describe particle diffraction from quantum periodic structures as the time independent Schrödinger equation for a harmonic potential is analytically identical to the corresponding Helmholtz equation describing optical diffraction from a harmonic index. Potential applications include the analysis of neutron super-mirrors that have been recorded using holographic techniques (13,14). Finally the results may also be useful in the study of acoustic diffraction from harmonic structures where the transfer matrix approach is well known (15).
13. M.Fally et al, Diffraction gratings for neutrons from polymers and holographic polymer-dispersed liquid crystals 2009 J. Opt. A: Pure Appl. Opt. 11 024019
14. J.Kleppetal,"HolographicGratingsforslow-neutron optics", Materials, 5, 2788- 2815;doi:10.3390/ma5122788 (2012)
15. M.Abid et al, "Acoustic Response of a Multilayer Panel with Viscoelastic Material", International Journal of Acoustics and Vibration, Vol. 17, No.2, pp82-89 (2012)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End
The PSM Theory is described in full detail in the follow references:
1) D. Brotherton-Ratcliffe, "A treatment of the general volume holographic grating as an array of parallel stacked mirrors," J. Mod. Optics. 59, 1113-1132 (2012).
2) D. Brotherton-Ratcliffe, "Analytical treatment of the polychromatic spatially multiplexed volume holographic grating," Appl. Opt. 51, 7188-7199 (2012).
3) D. Brotherton-Ratcliffe, "A new type of coupled wave theory capable of analytically describing diffraction in polychromatic spatially multiplexed holographic gratings," Journal of Physics, Conference Series 415 (2013) 012034 doi:10.1088/1742-6596/415/1/012034.
4) H. Bjelkhagen and D. Brotherton-Ratcliffe, Ultra- realistic imaging – advanced techniques in analogue and digital colour holography, (CRC- Taylor and Francis 2012).
5) D. Brotherton-Ratcliffe, "Understanding diffraction in volume gratings and holograms," in Holography - Basic Principles and Contemporary Applications, E. Mihaylova, ed., Chap. 1 (INTECH, 2013).
6) D. Brotherton-Ratcliffe, L. Shi. A. Osanlou and P. Excell, "A comparative study of the accuracy of the PSM and Kogelnik models of diffraction in reflection and transmission holographic gratings," Opt. Express 22 (26) 32384-32405 (2014).
7) D. Brotherton-Ratcliffe, A. Osanlou and P. Excell, "Using the parallel-stacked mirror model to analytically describe diffraction in the planar volume reflection grating with finite absorption", Applied Optics Vol. 54, pp. 3700–3707 (2015)
8) D. Brotherton-Ratcliffe, H. Bjelkhagen, A. Osanlou, and P. Excell, "Diffraction in volume reflection gratings with variable fringe contrast", Applied Optics Vol. 54, pp.5057-5064 (2015)
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