COMPUTER GENERATED HOLOGRAPHY
3D VISION AND BEYOND

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3D image
3D image
3D image
Photography

**Thin lens formula**

\[
\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}
\]
Photography

Original points reconstruction

- perfect for points in focus only
- loss of information
Hologram principle

Light diffraction
- depends mainly on frequency $f$ of the pattern
  output angle of the rays: $\sin \theta_{\text{out}} = m\lambda f + \sin \theta_{\text{in}}$

$m = +1$
$m = 0$
$m = -1$

$\theta_{\text{in}} = 0$

$\theta_{\text{out}}$

- low frequency pattern
- high frequency pattern
- diffracted rays
- undiffracted ray
Hologram watching

- illuminate hologram with a light source
- light beams start to diffract on the pattern as if the original object was still present
Nature of the light

- force interaction between (oscillating) point charges
- point source of a light: movement up and down $\sim A \cos(\omega t - \varphi)$
- force (field) in a distance $r$:
  $$u(t, r) = \frac{A}{r} \cos(\omega[t - \frac{r}{c}] - \varphi) = A' \cos(\omega t - \varphi'(r))$$
- photographic emulsion reacts on intensity: $(A')^2$
  $\Rightarrow$ cannot tell close “darker light” from distant “brighter light”
Interference

\[ \frac{A}{r_1} \cos(\omega [t - \frac{r_1}{c}]) + \frac{A}{r_2} \cos(\omega [t - \frac{r_2}{c}]) \]

- \( T \): period of oscillation \( \quad 1.7 \cdot 10^{-15} \) s
- \( \omega = \frac{2\pi}{T} \): angular frequency
- \( f = \frac{1}{T} \): frequency
- \( c \): speed of the light
- \( \lambda = cT \): wavelength \( \quad 0.5 \cdot 10^{-6} \) m
- \( k = \frac{2\pi}{\lambda} \): wavenumber \( \quad 1.2 \cdot 10^7 \) m\(^{-1} \)
Interference

\[
\frac{A}{r_1} \cos(\omega t - kr_1) + \frac{A}{r_2} \cos(\omega t - kr_2)
\]

\[
\approx 2 \frac{A}{r_1} \cos\left(\frac{k(r_1 - r_2)}{2}\right) \cos\left(\frac{2\omega t - k(r_1 + r_2)}{2}\right)
\]
Interference

- constructive
  ×
- destructive interference
Perfect picture

- image of $X$ in $\rho$: amplitude 0 except of $X'$ ($\rightarrow\infty$)
- image of $Y$ in $\rho'$: amplitude 0 except of $Y'$ ($\rightarrow\infty$)
- image of $Y$ in $\rho$: amplitude and phase from $Y'$
Perfect picture

- reconstruction of $X'$: point $X$
- reconstruction of "blurry" $Y'$: constructive interference in $Y'$
  $\Rightarrow$ reconstruction of $Y$
• phase is critical for 3D image – how to capture it?
• no need for a lens anymore
• observation from A: pseudoscopic image
• observation from B: orthoscopic image
Complex notation

- $j^2 = -1$
- $e^{jx} = \cos x + j \sin x$
- $A \cos(\omega t - \varphi) = \text{Re}\{A e^{j(\omega t - \varphi)}\}$

- $e^{jx} + e^{jy} = 2 \cos\left(\frac{x - y}{2}\right) \exp\left(j \frac{x + y}{2}\right)$
- $e^{jx} + e^{-jx} = 2 \cos x$

- Intensity of $U = A e^{j(\omega t - \varphi)}$
  
  $|U|^2 = U U^* = A e^{j(\omega t - \varphi)} A e^{-j(\omega t - \varphi)} = A^2$
Complex notation

Advantage of phasor arithmetic

- optical field – time dependent function:
  \[ u(t, r) = A \cos(\omega t - \varphi(r)) \]
- its phasor (complex amplitude):
  \[ U(r) = A \exp(-j\varphi(r)) \]
- sum of optical fields:
  \[ A_1 \cos(\omega t - \varphi_1 (r)) + A_2 \cos(\omega t - \varphi_2 (r)) + \ldots = ? \]
- in phasor arithmetic:
  \[ A_1 \exp(-j\varphi_1 (r)) + A_2 \exp(-j\varphi_2 (r)) + \ldots = U_{\text{sum}}(r) \]
- optical field (if needed):
  \[ u_{\text{sum}}(t, r) = \text{Re}\{U_{\text{sum}}(r) e^{j\omega t}\} \]
Basic wavefront shapes

**Spherical wavefront**

- \( u(t, r) = \frac{A}{r} \exp(j[\omega t - kr - \varphi]) \)
  
  \[ = \frac{A}{r} \exp(j\omega t) \exp(-j[kr + \varphi]) \]

- complex amplitude:
  \( U(r) = \frac{A}{r} \exp(-j[kr + \varphi]) \)

- resembles a plane in a big distance
Basic wavefront shapes

**Plane wavefront**

- wavefront normal $\mathbf{n}$, $|\mathbf{n}| = 1$
- wavefronts period $\lambda$
- wave vector $\mathbf{k} = k \mathbf{n} = 2\pi/\lambda \mathbf{n}$
- point in space $\mathbf{x} = (x, y, z)$
- wavefront plane equation $\mathbf{k} \cdot \mathbf{x} = \text{const.}$
- $U(\mathbf{x}) = A \exp(-j[k \cdot \mathbf{x} + \phi])$
- rays: “directions perpendicular to wavefronts”
Single slit diffraction

- complex amplitude at the slit of almost zero width
  \[ U(r_{in}) = \frac{A}{r_{in}} \exp(-j[kr_{in} + \varphi]) \]
- after “normalization”
  \[ U(0) = A' \]
- complex amplitude behind the opaque screen
  \[ U(r_{out}) = \frac{A'}{r_{out}} \exp(-j[kr_{out}]) \]
Multiple slit diffraction

- \( m \)-th diffraction maximum:
  \[
  \sin \theta_{\text{out}} = \frac{m \cdot \lambda}{d}
  \]
Multiple slit diffraction

• change of period ⇒ change of diffraction angles
• change of illumination angle (not shown)
  ⇒ \( \sin \theta_{\text{out}} = m \cdot \lambda / d + \sin \theta_{\text{in}} \) (grating equation)
Double slit diffraction

- screen in a plane $z = 0$
- two slits in a distance $d$
- angle of observation $\theta_{\text{out}}$
in a distance $r_{\text{out}} \gg d$
- change in $\theta_{\text{out}}$
  $\Rightarrow$ change in mutual "shift" of rays
  $\Rightarrow$ change of interference
Double slit diffraction

- in fact:
  - two point sources
  - common amplitude $A'$
  - phases $\varphi_1$, $\varphi_2$: 
    - $A'/r_{\text{out}} \exp(-j[kr_{\text{out}} + \varphi_1])$
    - $A'/r_{\text{out}} \exp(-j[kr_{\text{out}} + \varphi_0 + kd\sin \theta_{\text{out}}])$
  - their sum:
    - $2A'/r_{\text{out}} \cos \frac{\varphi_0 - \varphi_1 + kd\sin \theta_{\text{out}}}{2} \times$
    - $\times \exp(-j[kr_{\text{out}} + \frac{\varphi_0 + \varphi_1 + kd\sin \theta_{\text{out}}}{2}])$
Double slit diffraction

- $\pm 1$st diffraction maximum:
  \[
  \frac{\varphi_0 - \varphi_1 + kd \sin \theta_{\text{out}}}{2} = \pm \pi
  \]

- $m$-th diffraction maximum:
  \[
  \varphi_0 - \varphi_1 + kd \sin \theta_{\text{out}} = m \cdot 2\pi
  \]

$\varphi_0 = \varphi_1 = 0; \quad d = 3\lambda = 1500 \text{ nm}$
Double slit diffraction

- screen lighting by a plane wave at an angle $\theta_{in}$
- $\phi_1 = \phi_0 + kd \sin \theta_{in}$
- $m$-th diffraction maximum:
  $\phi_0 - \phi_1 + kd \sin \theta_{out} = m \cdot 2\pi$
- after substitution:
  $-kd \sin \theta_{in} + kd \sin \theta_{out} = m \cdot 2\pi$
  $\sin \theta_{out} = m\lambda/d + \sin \theta_{in}$

(grating equation)
Amplitude diffraction grating

- opaque screen with thin $N$ slits, period $d$

$$N\text{-thin-slit diffraction, } d = 3\lambda = 1.5 \, \mu m$$

Graph showing amplitude vs. $\theta_{out}$ for different values of $N$: 6, 4, 2, 1.
Amplitude diffraction grating

- other transmittance profiles:
  - different slit width
  - different transmittance shape
  ⇒ different brightness of maxima

- transmittance profile
  \[ \tau(x) = \frac{(1 + \cos(2\pi x/d))}{2} \]
  - the only important maxima:
  \[ m \in \{0, +1, -1\} \]
Cosine pattern diffraction

• plane wave $U(\mathbf{x}) = \exp(-j[\mathbf{k} \cdot \mathbf{x}])$ passing through a pattern with cosine transmittance profile:
  
  \[ U(\mathbf{x})|_{z=0} = \frac{1}{2} \left[ 1 + \cos(\frac{2\pi x}{d}) \right] \exp(-j[\mathbf{k} \cdot \mathbf{x}]) \]

  \[ = \frac{1}{2} \exp(-j[\mathbf{k} \cdot \mathbf{x}]) + \frac{1}{2} \cos(\frac{2\pi x}{d}) \exp(-j[\mathbf{k} \cdot \mathbf{x}]) \]

  \[ = \frac{1}{2} \exp(-j[\mathbf{k} \cdot \mathbf{x}]) \]

  \[ + \frac{1}{4} \left[ \exp(-j\frac{2\pi x}{d}) + \exp(j\frac{2\pi x}{d}) \right] \exp(-j[\mathbf{k} \cdot \mathbf{x}]) \]

  \[ = \frac{1}{2} \exp(-j[\mathbf{k} \cdot \mathbf{x}]) \]

  \[ + \frac{1}{4} \exp(-j[\mathbf{k}_+ \cdot \mathbf{x}]) + \frac{1}{4} \exp(-j[\mathbf{k}_- \cdot \mathbf{x}]) \]
Cosine profile recording

\[ d = \frac{\lambda}{\sin \theta_A - \sin \theta_B} \]

\[ f = \frac{1}{d} \]
Cosine profile recording

- plane wave complex amplitude: $A \exp(-j[k \cdot x])$
  inclination $\theta_A$: $k = k(\sin \theta_A, 0, \cos \theta_A)$
  in the plane $z = 0$: $x = (x, y, 0)$
  $\Rightarrow k \cdot x = kx \sin \theta_A$

- intensity of sum of plane waves from angles $\theta_A, \theta_B$:
  $|A \exp(-j k_A \cdot x) + A \exp(-j k_B \cdot x)|^2 =$
  $= (A \exp(-j k_A \cdot x) + A \exp(-j k_B \cdot x)) \times$
  $(A \exp(j k_A \cdot x) + A \exp(j k_B \cdot x)) =$
  $= 2A + 2A \cos(k_A \cdot x - k_B \cdot x) =$
  $= 2A \{1 + \cos(k[\sin \theta_A - \sin \theta_B]x)\}$

- frequency of the pattern $f = (\sin \theta_A - \sin \theta_B) / \lambda$
**Sin $\theta$ equation**

- grating equation:
  \[
  \sin \theta_{\text{out}} = m\lambda/d + \sin \theta_{\text{in}} = m\lambda f + \sin \theta_{\text{in}}
  \]

- after manipulation:
  \[
  \sin \theta_{\text{out}} = m(\sin \theta_A - \sin \theta_B) + \sin \theta_{\text{in}}
  \]

- example: $m = +1$, $\sin \theta_B = \sin \theta_{\text{in}}$
  \[
  \Rightarrow \sin \theta_{\text{out}} = \sin \theta_A
  \]
Hologram

- object wave: $\theta_{\text{obj}} \ (= \theta_A), \ \lambda = \lambda_{\text{ref}}$
- reference wave: $\theta_{\text{ref}} \ (= \theta_B), \ \lambda = \lambda_{\text{ref}}$
- illumination wave: $\theta_{\text{ill}} \ (= \theta_{\text{in}}), \ \lambda = \lambda_{\text{ill}}$
- $\sin \theta_{\text{out}} = m \frac{\lambda_{\text{ill}}}{\lambda_{\text{ref}}} (\sin \theta_{\text{obj}} - \sin \theta_{\text{ref}}) + \sin \theta_{\text{ill}}$
- example: $\lambda_{\text{ill}} = \lambda_{\text{ref}}, \ \theta_{\text{ill}} = \theta_{\text{ref}} = 0$

\[ \text{recording reconstruction} \]

- virtual image $m = +1$
- real image $m = -1$
Hologram recording
Hologram watching

**Light diffraction**
- depends mainly on frequency $f$ of the pattern
- output angle of the rays: $\sin \theta_{\text{out}} = m\lambda f + \sin \theta_{\text{in}}$

\[\theta_{\text{in}} = 0\]

- low frequency pattern: $m = -1$
- high frequency pattern: $m = 1$
- undiffracted ray: $m = 0$

\[\theta_{\text{out}}\]
Hologram watching

Virtual image creation

- illuminate hologram with a light source
- light beams start to diffract on the interference pattern as if the original object was still present

![Diagram showing light and hologram with diffracted rays and reconstruction light](image)
Real image creation

- output angle of the rays: \( \sin \theta_{\text{out}} = m \lambda f + \sin \theta_{\text{in}} \)
- for \( m = -1 \), rays create real image of the scene
- both rays for \( m = +1 \) and \( -1 \) appear at once
  \( \Rightarrow \) no need to distinguish between them
Hologram recording

Basic setups

- **in-line (Gabor) hologram**
  - for transparent objects
  - image damaged by 0th order
  - low spatial frequencies

- **off-axis (Leith-Upatnieks) hologram**
  - for opaque objects
  - clear image
  - high spatial frequencies (over 1000 lines/mm)
  - aberrations
Hologram principle proof

- hologram: recording of the interference of the object wave $O$ and the reference wave $R$:
  $$I = (O + R)(O + R)^* = OO^* + RR^* + OR^* + O^*R$$
- after illumination by the copy of the reference wave:
  $$U = IR$$
  $$= (OO^*)R \quad \text{diffracted illumination wave}$$
  $$+ (RR^*)R \quad \text{attenuated illumination wave}$$
  $$+ O(RR^*) \quad \text{copy of the object wave}$$
  $$+ O^*(RR) \quad \text{conjugate image}$$
3D display holography

- reconstruction wave (hologram illumination) the same as reference wave (in recording process)

⇒ observation of the original object

Hologram created by Šárka Němcová, ČVUT Praha
3D display holography

- reconstruction wave changes the angle
  ⇒ observation of the (deformed) original object from a varying viewpoint

Hologram created in The Central Laboratory of Optical Storage and Processing of Information, Bulgarian Academy of Sciences.
Holography applications

- microscopy, optical metrology
  - perfect light recording (biological sample, bubble chamber, ...)
  - hologram examination (unlimited time of observation, safe environment, ...)
- enhancing electron microscopy
  - original Gabor idea behind holography
  - hologram recording with electron beam
    (λ is 100 000× smaller than for visible light)
  - hologram enlargement, visible light illumination
    ⇒ image 100 000× bigger
Holography applications

- diffractive (holographic) optical elements
  - mimicking any optical element
  - cheaper, easier aberration correction, ...

![Diagram of holography process]

- light
- optical setup
- reference light
- "object" light
- hologram
- diffractive optical element recording

- light
- optical setup
- reconstruction light
- hologram
- diffractive optical element usage
- diffracted light

Computer generated holography: 3D vision and beyond

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Holography applications

- non-destructive testing
  - double object recording on one hologram
  - microshifts cause interference strips
  - vibration causes loss of interference strips

Molin and Stetson, Institute of Optical Research, Stockholm (1971)
Hologram creation mathematically for every point \((x, y)\) of the hologram:

- get the amplitude \(A_{\text{obj}}\) and the phase \(\phi_{\text{obj}}\) of the object wave in \((x, y)\)
- get the amplitude \(A_{\text{ref}}\) and the phase \(\phi_{\text{ref}}\) of the reference wave in \((x, y)\)
- calculate captured intensity in \((x, y)\)
  \[
  I(x, y) = |A_{\text{obj}} \exp(-j \phi_{\text{obj}}) + A_{\text{ref}} \exp(-j \phi_{\text{ref}})|^2
  \]
Object wave

Complex amplitude of a point source

- point source $P$ at $(x_p, y_p, z_p)$, $z_p < 0$
- light amplitude $A_p$, phase $\varphi_p$, wavelength $\lambda \approx 630$ nm ($\Rightarrow k = \frac{2\pi}{\lambda} \approx 10^7$)
- hologram plane $z = 0$

- $U_p(x, y, 0) = \frac{A_p}{r_p} \exp(-j[kr_p + \varphi_p])$
  
  $r_p = [(x - x_p)^2 + (y - y_p)^2 + z_p^2]^{1/2}$
Really unoptimized Matlab code

```matlab
lambda     = 630e-9;
k          = 2*pi/lambda;
res_x      = 200;
res_y      = 200;
hologram_z = 0;
sampling   = 20e-6;
corner_x   = -(res_x-1) * sampling / 2;
corner_y   = -(res_y-1) * sampling / 2;
sources    = [0, 0, -0.5; 20*sampling, 0, -0.5; -40*sampling, 20*sampling, -0.5];
```
Object wave

```
objectwave = zeros(res_y, res_x);
for source = 1:rows(sources)
    for column = 1:res_x
        for row = 1:res_y
            x = (column-1) * sampling + corner_x;
            y = (row-1) * sampling + corner_y;
            objectwave(row,column) +=
                exp(i*k*sqrt((x-sources(source, 1))**2
                            + (y-sources(source, 2))**2
                            + (hologram_z - sources(source, 3))**2));
        endfor
    endfor
endfor
```
Object wave

Real part of the object wave
(Just for information;
it has no physical meaning!)
Reference wave

Complex amplitude of a reference wave

- plane wave with direction vector $\mathbf{n}_R = (n_{Rx}, n_{ Ry}, n_{Rz})$, $|\mathbf{n}_R| = 1$
- let us ignore constant phase ($\Rightarrow \varphi = 0$)
- $U_R(x, y, 0) = A_R \exp(-j[k \mathbf{n}_R \cdot \mathbf{x} + \varphi]) = A_R \exp(-jk[xn_{Rx} + yn_{Ry}])$
Reference wave

```matlab
refwave = zeros(res_y, res_x);
ref_x = cos(89.9 * pi/180) * k;
ref_y = cos(90 * pi/180) * k;

for column = 1:res_x
    for row = 1:res_y
        x = (column-1) * sampling + corner_x;
        y = (row-1) * sampling + corner_y;
        refwave(row,column) = exp(i*(ref_x * x + ref_y * y));
    endfor
endfor
```
Reference wave

Real part of the reference wave
(Just for information; it has no physical meaning!)
Hologram calculation

Intensity calculation

- \( I(x, y, 0) = |U_R(x, y, 0) + U_P(x, y, 0)|^2 \)
  
  \[ = \left[ U_R(x, y, 0) + U_P(x, y, 0) \right] \times \]
  
  \[ \times \left[ U_R(x, y, 0) + U_P(x, y, 0) \right]^* \]

\[ = U_R U_R^* + U_P U_P^* + U_R U_P^* + U_P U_R^* \]

\[ a \quad b \quad c \]

**a)** reference wave intensity

**b)** object points interference (if \( U_P \) is a complex wave)

**c)** object points and reference wave interference
   (bipolar intensity)
Hologram calculation

\[
\text{hologram} = \text{objectwave} + \text{refwave};
\]
\[
\text{hologram} = \text{hologram} \times \text{conj(hologram)};
\]

- alternative (bipolar intensity):

\[
\text{hologram} = \text{real(objectwave)} \times \text{real(refwave)} + \\
\text{imag(objectwave)} \times \text{imag(refwave)}
\]
Hologram calculation

Object wave

Reference wave

The hologram (intensity picture)
Hologram calculation

Computer generated hologram

6144 × 6144 pixels
Size 4.3 × 4.3 cm²
(resolution 3600 dpi
≈ pixel size 7 μm)
Static high resolution holograms

- electron beam lithography
  - 0.1 μm details
  - ⇒ diffraction up to 90°
  - extremely expensive,
    recording 1 mm²/min
- laser lithography
  - 1 μm details
  - ⇒ diffraction up to 20°
  - very expensive,
    recording 4 mm²/min

Hologram by K. Matsushima
Home made static holograms

- imagesetter
  - 10 μm details
    ⇒ diffraction up to 2°
  - price ~ 5 € per A4
- laser printer
  - 100 μm details
    ⇒ diffraction up to 0.5°

Hologram by I. Hanáč, M. Janda
Hologram portrayal

Laboratory holographic displays

- based on DMD chips (DLP projectors), phase only spatial light modulators or acousto-optic modulators: (Bilkent University, MIT Media Lab, ...)
- based on intermediate optical photorefractive memory (University of Arizona)

DMD chip by Texas Instruments
Hologram portrayal

**Early stage commercial displays**

- Zebra Imaging
- SeeReal Technologies
  - spatial light modulators plus eye tracking
- QinetiQ
  - spatial light modulator plus intermediate optical memory

SeeReal Visio 20”

Zebra Imaging ZScape motion display
Digital holographic microscopy

- acquisition of digital hologram
- numerical reconstruction

⇒ signal filtering, unwanted diffraction removal, numerical analysis, ...
Digital holography applications

Surface metrology

- real object numerical reconstruction
- reconstructed phase ~ surface bumpiness

\[ \text{laser} \rightarrow \text{pinhole lens} \rightarrow \text{lens} \rightarrow \text{splitter} \rightarrow \text{sample} \rightarrow \text{CCD} \]

\[ \text{mirror} \]

\[ \text{captured phase} \rightarrow \text{unwrapped phase (Jüptner, Schnars: Digital Holography)} \]
Comparative digital holography

- hologram of master sample (A)
- reconstruction of master to object B

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**Digital holography applications**

- laser
- lens
- pinhole
- observer
- splitter
- master
- real image
- SLM
- captured phase
- unwrapped phase  
  (Schnars: Digital Holography)
Digital holography applications

Signal processing

- conversion between plane and spherical wave: convex lens of focal length $f$

![Diagram showing signal processing with point sources and focused points through a lens with focal length $f$.]
Digital holography applications

Signal processing

- complex amplitude of plane wave at plane $z = 0$:
  \[ U(x, y) = A_{ab} \exp(-j[ax + by]) \]
- $a, b$ depend on wave inclination
- illumination with many plane waves:
  \[ U(x, y) = \int_a \int_b A_{ab} \exp(-j[ax + by]) \, da \, db \]

$\Rightarrow$ can be considered as Fourier transform of $A_{ab}$

- Fourier transform (not a proper definition!):
  \[ \text{FT}\{A(a, b)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A_{ab} \exp(-j[ax + by]) \, da \, db \]
Digital holography applications

Signal processing

- 2f system – optical Fourier transform unit
- 4f system – optical filtering system
Holographic memory

**Multiple exposure of single hologram**

**Selective reconstruction by reconstruction wave change**
Holographic memory

- spatial light modulator (SLM) A: data
- SLM B: address

**Diagram:**
- A laser splits into two beams.
- One beam passes through SLM A to form a hologram.
- The other beam passes through SLM B, forming another hologram.
- Multiple exposure of a single hologram occurs.
- Selective reconstruction by reconstruction wave change.
Holographic cryptography

- SLM A: data, SLM B: key
- wrong key reconstruction: scrambled output

**Diagram:**

- **Encryption**
  - Laser → Splitter → SLM A → Hologram
  - SLM B

- **Correct Key Reconstruction**
  - Laser → Splitter → SLM B → Hologram

- **Wrong Key Reconstruction**
  - Laser → Splitter → SLM B (scrambled pattern)
Rayleigh-Sommerfeld integral

\[ U(x, y, z_0) = -\frac{1}{2\pi} \iint_{\text{hologram}} U(\xi, \eta, 0) \times \]
\[ \times (-jkr - \frac{1}{r}) \frac{\exp(-jkr) z_0}{r} \, d\xi \, d\eta \]

\[ r = [(x - \xi)^2 + (y - \eta)^2 + z_0^2]^{1/2} \]
Numerical propagation

Discrete calculation

- discretization of areas to $M \times N$ samples
- samples distance $\Delta$
- $x = (m - M/2) \Delta, \ y = (n - N/2) \Delta$

\[
U'[p, q] = -\frac{1}{2\pi} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U[m, n] \ K[p - m, q - n]
\]

\[
K[p, q] = (-jk - \frac{1}{r}) \frac{\exp(-jkr) z_0}{r^3}
\]

\[
r = [(p\Delta)^2 + (q\Delta)^2 + z_0^2]^{1/2}
\]
Numerical propagation

Discrete calculation

\[ U'[p, q] = -\frac{1}{2\pi} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U[m, n] K[p - m, q - n] \]

- \( p = 0, m = M - 1; \quad q = 0, n = N - 1 \)
  \( \Rightarrow \) minimal indices \( K: -(M - 1), -(N - 1) \)
- \( p = M - 1, m = 0; \quad q = N - 1, n = 0 \)
  \( \Rightarrow \) maximal indices \( K: +(M - 1), +(N - 1) \)

- \( K \) has to be known in \((2M - 1) \times (2N - 1)\) samples
Numerical propagation

Discrete cyclic convolution

- padding $U[m, n]$ with zeros to $(2M - 1) \times (2N - 1)$

- $U'[p, q] = -\frac{1}{2\pi} \sum_{m=0}^{2M-2} \sum_{n=0}^{2N-2} U[m, n] \times$
  
  \[ \times K_c[p - m \pmod{2M - 1}, q - n \pmod{2N - 1}] \]

  $= -\frac{1}{2\pi} \text{IDFT}\{\text{DFT}(U) \circ \text{DFT}(K)\}$

**DFT**  discrete Fourier transform

**IDFT** inverse discrete Fourier transform

**⊙** element-by-element multiplication
Discrete cyclic convolution

- example for $M = N = 4$

Structure of $K$

<table>
<thead>
<tr>
<th>Original Index of $K$</th>
<th>(New) Index of $K_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3, -2, -1, 0, +1, +2, +3</td>
<td>0, -1, 1, 2, 3, -2, -1, 6</td>
</tr>
</tbody>
</table>

Structure of $K_c$
Numerical propagation

propag_z = -0.5;

kernel = zeros(2*res_y, 2*res_x);
if (propag_z < 0) ii = -i; else ii = i; endif

for column = 1:2*res_x
  for row = 1:2*res_y
    if (column < res_x)
      x = (column-1) * sampling;
    else
      x = (column-2*res_x-1) * sampling;
    endif
if (row < res_y)
    y = (row-1) * sampling;
else
    y = (row-2*res_y-1) * sampling;
endif

r2 = x**2 + y**2 + propag_z**2;
kernel(row,column) =
    ii * k * exp(ii*k*sqrt(r2)) / r2;
endfor
endfor
Numerical propagation

Real part of the Rayleigh-Sommerfeld cyclic convolution kernel (Just for information; it has no physical meaning!)
Numerical propagation

\[
\text{field} = \text{zeros}(2*\text{res}_y, 2*\text{res}_x);
\text{field}(1:\text{res}_y, 1:\text{res}_x) = \text{hologram};
\]

\[
\text{FTfield} = \text{fft2}(\text{field});
\text{FTkernel} = \text{fft2}(\text{kernel});
\]

\[
\text{FTfield2} = \text{FTfield} .* \text{FTkernel};
\]

\[
\text{field2} = \text{ifft2}(\text{FTfield2});
\text{image} = \text{field2}(1:\text{res}_y, 1:\text{res}_x);
\]
Numerical propagation

Numerical simulation of hologram propagation
(Intensity picture – this would be actually captured)
Numerical propagation

Optical reconstruction

Numerical reconstruction
Numerical propagation

- **forward propagation**
  - in the $z^+$ axis direction
  - hologram propagation – in a distance $z_0 > 0$
    - real image appears – on-screen projection
  - original complex field propagation
    $U_p(x, y, 0)$ – no real image on $z^+$ axis

- **backward propagation**
  - propagation to a distance $z_0 < 0$
  - convolution kernel $K_c$ has to be complex conjugate
Numerical propagation

Lens simulation

1. propagation to a distance $r$: phase shift $kr$

2. propagation in a lens: phase shift $\varphi$

3. propagation to a distance $r'$: phase shift $kr'$

• all contributions in phase in point $X'$

$\Rightarrow$ phase function of a lens $\varphi = -(kr + kr')$

$\Rightarrow$ in $(x, y, 0)$: $\varphi = -k[(x^2 + y^2 + a^2)^{1/2} + (x^2 + y^2 + a'^2)^{1/2}]$

$\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}$
Hologram of a 3D scene

- object replacement with a point cloud
  - extraordinary number of lights needed ⇒ slow
  - does not count with visibility
  - easy parallelization ⇒ fast for thousands of points
Hologram of a 3D scene

- object replacement with a flat image
  - the same as hologram propagation – use of DFT
  - not for a 3D scene
Hologram of a 3D scene

- object replacement with series of flat images
  - propagation $A \rightarrow H$, $B \rightarrow H$, $C \rightarrow H$, sum
  - simulation of 3D scene, use of DFT
  - does not count with visibility
Hologram of a 3D scene

- step-by-step propagation
  - propagation A→B, masking,
    B→C, masking, C→H
  - enables to replace 3D scene with several slices
Hologram of a 3D scene

- general step-by-step propagation
  - rotation $A \rightarrow A'$, propagation $A' \rightarrow B'$, rotation $B' \rightarrow B$, masking, rotation $B \rightarrow B'$, propagation $B' \rightarrow C'$, ...
  - enables to render a scene with textured polygons
Hologram of a 3D scene

- point cloud rendering enhanced with ray casting for visibility testing
  - extremely slow

\[\begin{align*}
\text{invisible} & \Rightarrow \text{does not contribute} \\
\text{visible} & \Rightarrow \text{contributes}
\end{align*}\]
Hologram of a 3D scene

- scene breakup to rectangular patches
  - common visibility solution for a number of point sources and a number of hologram points

\[
\text{scene approximation}
\]

\[
\text{visible } \Rightarrow \text{ contributes}
\]

\[
\text{invisible } \Rightarrow \text{ does not contribute}
\]
Hologram of a 3D scene

- analytic triangle patch propagation formula
  - visibility solution in one view only (mostly)
  - problem with diffuse surface reflection
- analytic line propagation formula
  - for wireframe models
Hologram of a 3D scene

- precalculated table of point sources fields, their fast summation on GPU
- approximation of light propagation
  - Rayleigh-Sommerfeld convolution $3 \times$ DFT
  - angular spectrum decomposition $2 \times$ DFT, direct calculation of DFT(kernel)
  - Fresnel approximation $1 \times$ DFT, paraxial
  - Fraunhofer approximation $1 \times$ DFT, paraxial, big distances
Angular spectrum decomposition

- a plane wave hitting plane $z = 0$:
  \[ U(x, y, 0) = A \exp\{-jk(ax + by)\}\]
  propagation vector \( \mathbf{n} = (a, b, [1 - a^2 - b^2]^{1/2}) \)
  \[ a = \mathbf{n} \cdot (1, 0, 0) = \cos \theta_x \]
  \[ b = \mathbf{n} \cdot (0, 1, 0) = \cos \theta_y \]
  \[ \{ \text{direction cosines} \]  

- many plane waves hitting plane $z = 0$:
  \[ U(x, y, 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(a/\lambda, b/\lambda) \exp\{-jk(ax + by)\} \, da \, db \]
  with \( A(a/\lambda, b/\lambda) = 0 \) for \(|a| > 1, |b| > 1\)  
  - definition of \( A(a/\lambda, b/\lambda) \) instead of clearer \( A(a, b) \)
  will be advantageous in a while
Angular spectrum decomposition

- more often: \( f_x = a/\lambda, f_y = b/\lambda \)
  
i.e.
  \[
  U(x, y, 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(f_x, f_y) \exp\{-2\pi j (f_x x + f_y y)\} \, df_x \, df_y
  
  = \text{FT}\{ A(f_x, f_y) \}
  
  i.e.
  \[
  A(f_x, f_y) = \text{FT}^{-1}\{ U(x, y, 0) \}
  
  = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U(x, y, 0) \exp\{2\pi j (f_x x + f_y y)\} \, dx \, dy
  \]
Angular spectrum decomposition

- a plane wave hitting plane $z = z_0$:
  \[ U(x, y, z_0) = A \exp\{-jk(ax + by + cz_0)\} \]
  \[ = A \exp\{-jk(ax + by)\} \exp\{-j kz_0 c\} \]
  \[ = A \exp\{-jk(ax + by)\} \exp\{-j kz_0 [1 - a^2 - b^2]^{1/2}\}\]

- many planes hitting plane $z = z_0$:
  \[ U(x, y, z_0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(f_x, f_y) \exp\{-j2\pi z_0 [1/\lambda^2 - f_x^2 + f_y^2]^{1/2}\}\]
  \[ \exp\{-j2\pi (f_x x + f_y y)\} \, df_x \, df_y \]
  \[ = \text{FT}\{ A(f_x, f_y) \exp\{-j2\pi z_0 [1/\lambda^2 - f_x^2 + f_y^2]^{1/2}\}\}\]
Angular spectrum propagation

input: \( U(x, y, 0) \)
output: \( U(x, y, z_0) \)
calculation:
\[
A(f_x, f_y) = \text{FT}^{-1}\{ U(x, y, 0) \} \\
U(x, y, z_0) = \text{FT}\{ A(f_x, f_y) \} \\
\quad \exp\{-j2\pi z_0 [1/\lambda^2 - f_x^2 + f_y^2]^{1/2}\}\}
\]

- mathematically equivalent to the R-S convolution
- just two Fourier transforms
- numerically easier for small \( z_0 \)
  (R-S is better for bigger \( z_0 \) – see kernel sampling)
Fresnel approximation

Rayleigh-Sommerfeld solution

\[ U(x, y, z_0) = -\frac{1}{2\pi} \int \int_{\text{hologram}} U(\xi, \eta, 0) \times \]
\[ \times (-jk - \frac{1}{r}) \frac{\exp(-jkr) z_0}{r} \frac{1}{r} \ d\xi \ d\eta \]

\[ r = [(x - \xi)^2 + (y - \eta)^2 + z_0^2]^{1/2} \]
\[ = z_0 [1 + (x - \xi)^2/z_0^2 + (y - \eta)^2/z_0^2]^{1/2} \]
\[ \div z_0 [1 + (x - \xi)^2/2z_0^2 + (y - \eta)^2/2z_0^2] \]
\[ = z_0 + (x - \xi)^2/2z_0 + (y - \eta)^2/2z_0 \]
\[ = z_0 + (x^2 + y^2)/2z_0 + (\xi^2 + \eta^2)/2z_0 - (x\xi + y\eta)/z_0 \]
Fresnel approximation

For $z_0 \gg x, y$:

$$U(x, y, z_0) = -\frac{1}{2\pi} \iint_{\text{hologram}} U(\xi, \eta, 0) \times$$

$$\times (-jk - \frac{1}{r}) \frac{\exp(-jkr) z_0}{r} \frac{1}{r} \, d\xi \, d\eta$$

$$= \frac{j k z_0}{2\pi} \iint_{\text{hologram}} U(\xi, \eta, 0) \frac{\exp(-jkr)}{r^2} \, d\xi \, d\eta$$

$$= \frac{j k}{2\pi z_0} \iint_{\text{hologram}} U(\xi, \eta, 0) \exp(-jkr) \, d\xi \, d\eta$$
Fresnel approximation

\[
\frac{jk}{2\pi z_0} \iint_{\text{hologram}} U(\xi, \eta, 0) \times \exp(-jk[z_0 + (x^2 + y^2)/2z_0 + \frac{(\xi^2 + \eta^2)/2z_0 - (x\xi + y\eta)/z_0)]/\lambda z_0) \, d\xi \, d\eta
\]

\[
= \frac{jk}{2\pi z_0} \exp(-jkz_0) \exp(-jk(x^2 + y^2)/2z_0) \times \iint_{\text{hologram}} U(\xi, \eta, 0) \exp(-jk(\xi^2 + \eta^2)/2z_0) \times \exp(-j2\pi(x\xi + y\eta)/\lambda z_0) \, d\xi \, d\eta
\]
Fresnel approximation

\[
\frac{jk}{2\pi z_0} \exp(-jkz_0) \exp(-jk(x^2 + y^2)/2z_0) \times \\
\text{FT}\{ U(\xi, \eta, 0) \exp(-jk(\xi^2 + \eta^2)/2z_0) \}
\]

where after FT calculation substitute
\[
f_x = x/\lambda z_0 \\
f_y = y/\lambda z_0
\]

• approximation valid for on-axis case, big \( z_0 \)
  \[z_0^3 \gg \pi/4\lambda \max\{(x - \xi)^2 + (y - \eta)^2\}^2\]
• just one Fourier transform
Hologram of a 3D scene

- classical H1 – H2 process
  1. make a classical hologram (H1)
Hologram of a 3D scene

- classical H1 – H2 process
  2. illuminate H1 with a conjugate wave
  3. make a hologram of a hologram (H2)
Hologram of a 3D scene

- classical H1 – H2 process
  4. illuminate H2 with a conjugate wave
     – an orthoscopic image, viewing aperture H1
Hologram of a 3D scene

- classical white light hologram
  - H1 – hologram of a scene
    “viewed through a narrow window”
- digitally: slow calculation, small H1 surface
Hologram of a 3D scene

- classical white light hologram
  - $H_2$ – hologram of the $H_1$ hologram
- digitally: no visibility checks $\Rightarrow$ fast calculation
Hologram of a 3D scene

- classical white light hologram reconstruction
  - resembles view through a narrow window
  - horizontal parallax only image

reconstructed virtual object

reconstruction light

H2

reconstructed "window"
Hologram of a 3D scene

- classical white light hologram reconstruction
  - “wrong” reconstruction color shifts reconstruction
  - H2 extracts “the right” color from white light
Hologram of a 3D scene

- white light hologram structure
  - just “bold” points will be visible due to rays in the cutting plane
Hologram of a 3D scene

- white light hologram structure
  - in H2 recording, those “bold” point will affect only a part of the H2
  ⇒ “bold” points affect a part of H2 only
Hologram of a 3D scene

- digital white light HPO hologram (1)
  - split the H2 into parts – hololines
  - just one line of the hololine is considered
  - calculate the hololine using “bold” points only
Hologram of a 3D scene

- digital white light HPO hologram (2)
  - assume the “bold” points to be lines
  - they emit cylindrical wave
  - object wave constant in vertical direction
Hologram of a 3D scene

- digital white light HPO hologram (3)
  - hololine has the area \( \text{width} \times \text{height} \)
  - object wave in every horizontal line (subline) is the same \( \Rightarrow \) calculate once & copy
Hologram of a 3D scene

• digital white light HPO hologram (4)
  – in reconstruction, the wave from a hololine should hit the observation window
Hologram of a 3D scene

- digital white light HPO hologram (4)
  - the light has to change its angle from $\theta_{ill}$ to $\theta_{out N}$
  - in hololine $N$, add reference wave with angle
    \[ \sin \theta_{ref} = \sin \theta_{ill} - \sin \theta_{out N} \]
Hologram of a 3D scene

- classical holographic stereogram (1)
  - record left image to the left part of H1 only

reference light
Hologram of a 3D scene

- classical holographic stereogram (2)
  - record right image to the right part of H1 only
Hologram of a 3D scene

- classical holographic stereogram (3)
  - record H2 in a common way
Hologram of a 3D scene

- classical holographic stereogram (4)
  - after illumination, the left eye watches the left image, the right eye watches the right image
Hologram of a 3D scene

- digital holographic stereogram
  - visibility solving in particular directions using computer graphics (ray optics)
  - hologram has to display right image in the right direction
  - compatible with common imaging cameras
Questions?

- graphics.zcu.cz
- holo.zcu.cz
- www.kiv.zcu.cz